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ON DIOPHANTINE APPROXIMATION AND THE GENERALIZED ELLIPTIC THETA FUNCTION

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[Received 1 November 1937]

1. Introduction

LET $Q(\mathfrak{x}) = \sum_{j=1}^{m} \sum_{k=1}^{m} s_{jk} x_j x_k$ be a positive definite quadratic form with integer coefficients and

$$f(\tau) = \sum_{\mathfrak{x}} e^{\pi i \tau Q(\mathfrak{x})} \quad (\tau = x+iy;\ y > 0), \tag{1}$$

where the summation is over all integer values of $x_1,..., x_m$. It is well known that the elliptic theta function $f(\tau)$ defined by this series is analytic throughout the half-plane $y > 0$ and that the line $y = 0$ is a natural boundary of the function. We shall investigate the asymptotic behaviour of $f(\tau)$ as τ moves up to various types of points on the natural boundary.

Let $A(n)$ denote the number of integer solutions of

$$Q(\mathfrak{x}) = n.$$

Then, as the series (1) is absolutely convergent for $y > 0$, we may write it in the form

$$f(\tau) = 1+\sum_{n=1}^{\infty} A(n) e^{\pi i \tau n}. \tag{2}$$

This is the Abel sum of the trigonometric sum

$$\begin{aligned} R(N,x) &= \sum_{n=0}^{N} A(n) e^{\pi i x n} \\ &= \sum_{Q(\mathfrak{x}) \leqslant N} e^{\pi i x Q(\mathfrak{x})}, \end{aligned} \tag{3}$$

whose asymptotic behaviour we shall also consider.

The solution by Hardy and Littlewood* of the corresponding problem for

$$\vartheta_3(0|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i \tau n^2}$$

depends on the transformation formula†

$$\vartheta_3(0|\tau) = \kappa(c\tau+d)^{-\frac{1}{2}} \vartheta_3\left(0 \left| \frac{a\tau+b}{c\tau+d}\right.\right) \quad (\kappa \text{ a constant}),$$

* Hardy and Littlewood, **1**. † See Hardy and Littlewood, **1**, p. 227.

which holds for transformations of the modular group. Siegel* has proved a similar formula for $f(\tau)$ but only for the sub-group of the modular group satisfying the congruence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2S},$$

where S denotes the determinant of the matrix $\mathfrak{S} = (s_{jk})$. This condition makes it useless for our purpose. We shall prove a formula (see Lemma 4) less precise than Siegel's, but sufficient for our purpose and free from such conditions. Otherwise, the proof follows the method of Hardy and Littlewood.

2. Notation

We shall adopt the notation of Siegel. If $\mathfrak{A}$ is a matrix, let $\mathfrak{A}'$ denote the transpose of $\mathfrak{A}$, i.e. the matrix obtained from $\mathfrak{A}$ by interchanging rows and columns. Thus $\mathfrak{S}' = \mathfrak{S}$ for the symmetric matrix $\mathfrak{S}$. In the case when $\mathfrak{A}$ is a non-singular square matrix we denote its inverse by $\mathfrak{A}^{-1}$. The column of elements $x_1, x_2,\ldots, x_m$ will be denoted by the corresponding small German letter $\mathfrak{x}$, so that $\mathfrak{x}'$ will be the *row* of elements $(x_1,\ldots, x_m)$. We shall call a matrix an *integer* when all its elements are integers. If $\mathfrak{U}$ is an integer matrix whose determinant $|\mathfrak{U}| = 1$, we shall call it a *unit* matrix. We shall write

$$\mathfrak{A} \equiv \mathfrak{B} \pmod q$$

when all the elements of $\mathfrak{A}-\mathfrak{B}$ are divisible by q.

To indicate a summation over all integer columns $\mathfrak{x}$ we shall write $\sum\limits_{\mathfrak{x}}$, while $\sum\limits_{\mathfrak{x}}{}'$ will mean such a summation with the null column $\mathfrak{n}$ omitted. For a summation over a complete remainder system to modulus b we shall write $\sum\limits_{\mathfrak{x}(b)}$ (i.e. each element of $\mathfrak{x}$ runs over a complete remainder system to modulus b). If $\mathfrak{x}$ runs over all those integer columns of a complete remainder system to modulus b which satisfy the congruence $\mathfrak{A}x \equiv \mathfrak{n} \pmod\beta$, where $\beta \mid b$, we shall write $\sum\limits_{\substack{\mathfrak{x}(b) \\ \mathfrak{A}\mathfrak{x}\equiv\mathfrak{n}(\beta)}}$.

We reserve $\mathfrak{S}$ to denote the positive definite symmetric integer matrix (s_{jk}) of m rows and columns and determinant $S = |\mathfrak{S}|$. Then the quadratic form under discussion may be written

$$Q(\mathfrak{x}) = \mathfrak{x}'\mathfrak{S}\mathfrak{x}.$$

If all the elements of the principal diagonal of $\mathfrak{S}$ are even, then $Q(\mathfrak{x})$

* Siegel, **1**, Lemma 30.

can only represent even integers. We therefore describe $\mathfrak{S}$ as even in this case; otherwise we say that $\mathfrak{S}$ is odd.

Let d_k be the greatest common divisor of all the sub-determinants of $\mathfrak{S}$ of the kth order, so that $d_m = |\mathfrak{S}|$. Write

$$e_1 = d_1, \qquad e_k = d_k/d_{k-1} \quad (k = 2, 3, \dots, m)$$

Then the numbers $e_1, e_2, \dots, e_m$, which may easily be shown to be integers, are called the elementary divisors of $\mathfrak{S}$.

3. Congruences

The central theorem in the theory* of elementary divisors is

LEMMA 1. *There exist two unit matrices* $\mathfrak{U}$ *and* $\mathfrak{V}$ *such that*

$$\mathfrak{U}^{-1}\mathfrak{S}\mathfrak{V} = \mathfrak{D},$$

where $\mathfrak{D}$ *is the diagonal matrix* $(e_i \delta_{ij})$, δ_{ij} *being Kronecker's symbol.*

An easy deduction from Lemma 1 is

LEMMA 2.† *Let*
$$N(\mathfrak{S}, b) = \sum_{\substack{\mathfrak{x}(b) \\ \mathfrak{S}\mathfrak{x} \equiv \mathfrak{n}(b)}} 1,$$

i.e. $N(\mathfrak{S}, b)$ *is the number of integer solutions of*

$$\mathfrak{S}\mathfrak{x} \equiv \mathfrak{n} \pmod b$$

which are incongruent to modulus b. *Then*

$$N(\mathfrak{S}, b) = s_1 s_2 \dots s_m,$$

where $s_k = (e_k, b)$.

We note that, as $s_k \leqslant e_k$,

$$N(\mathfrak{S}, b) \leqslant e_1 e_2 \dots e_m = d_m = S. \tag{4}$$

Also, if b is odd, the congruence $2\mathfrak{S}\mathfrak{x} \equiv \mathfrak{n} \pmod b$ implies $\mathfrak{S}\mathfrak{x} \equiv \mathfrak{n} \pmod b$ so that

$$N(2\mathfrak{S}, b) = N(\mathfrak{S}, b) \quad (b \text{ odd}). \tag{5}$$

If b is even and $b = 2\beta$, the congruence $2\mathfrak{S}\mathfrak{x} \equiv \mathfrak{n} \pmod b$ implies $\mathfrak{S}\mathfrak{x} \equiv \mathfrak{n} \pmod \beta$ so that, using the substitution $\mathfrak{x} = \beta\mathfrak{y} + \mathfrak{z}$,

$$N(2\mathfrak{S}, b) = \sum_{\substack{\mathfrak{x}(b) \\ \mathfrak{S}\mathfrak{x} \equiv \mathfrak{n}(\beta)}} 1 = \sum_{\substack{\mathfrak{z}(\beta) \\ \mathfrak{S}\mathfrak{z} \equiv \mathfrak{n}(\beta)}} \sum_{\mathfrak{y}(2)} 1 = 2^m N(\mathfrak{S}, \beta). \tag{6}$$

In either case we have

$$N(2\mathfrak{S}, b) \leqslant 2^m S. \tag{7}$$

* See H. J. S. Smith **1**, 391, or Bachmann, **1**.

† See e.g. H. J. S. Smith, **1**, 399.

4. The generalized Gaussian sum

Let us now consider the generalized Gaussian sum

$$G(\mathfrak{S}, a/b, \mathfrak{h}) = \sum_{\mathfrak{x}(b)} e^{2\pi i\{(a/b)\mathfrak{x}'\mathfrak{S}\mathfrak{x}+\mathfrak{x}'\mathfrak{h}\}},$$

where $(a,b)=1$, $b>0$, and $\mathfrak{h}$ is a column of m rational elements such that $b\mathfrak{h}$ is an integer column. When $\mathfrak{h}=\mathfrak{n}$, the null column, we write

$$G(\mathfrak{S}, a/b, \mathfrak{n}) = G(\mathfrak{S}, a/b).$$

The following lemma is merely a refinement on that of Siegel.*

Lemma 3. *We have*

$$|G(\mathfrak{S}, a/b, \mathfrak{h})| \leqslant b^{m/2}\{N(2\mathfrak{S}, b)\}^{\frac{1}{2}} \leqslant 2^{m/2}S^{\frac{1}{2}}b^{m/2},$$

and

$$|G(\mathfrak{S}, a/b)| = \begin{cases} b^{m/2}\{N(\mathfrak{S}, b)\}^{\frac{1}{2}} & (b \text{ odd}), \\ 0 & (b=2\beta;\ \beta \text{ odd and } \mathfrak{S} \text{ odd}), \\ (2b)^{m/2}\{N(\mathfrak{S}, \beta)\}^{\frac{1}{2}} & (b=2\beta;\ \beta \text{ odd and } \mathfrak{S} \text{ even}), \\ 0 & (b=2\beta;\ \beta \text{ even and } \mathfrak{C} \text{ odd}), \\ (2b)^{m/2}\{N(\mathfrak{S}, \beta)\}^{\frac{1}{2}} & (b=2\beta;\ \beta \text{ even and } \mathfrak{C} \text{ even}), \end{cases}$$

where

$$\mathfrak{C} = (c_{jk}) = \left(\frac{\beta e_k}{\sigma_j\sigma_k} w_{jk}\right), \qquad \mathfrak{W} = (w_{jk}) = \mathfrak{V}'\mathfrak{U}, \qquad \sigma_k = (e_k, \beta).$$

The matrix $\mathfrak{C}$ is clearly an integer. By Lemma 1

$$\mathfrak{S}\mathfrak{V} = \mathfrak{U}\mathfrak{D},$$

so that $$\mathfrak{W}\mathfrak{D} = \mathfrak{V}'\mathfrak{U}\mathfrak{D} = \mathfrak{V}'\mathfrak{S}\mathfrak{V} = (\mathfrak{V}'\mathfrak{S}\mathfrak{V})' = (\mathfrak{W}\mathfrak{D})',$$

i.e. $\mathfrak{W}\mathfrak{D}$ $(=(w_{jk}e_k))$ is symmetric. Hence $\mathfrak{C}$ is symmetric. Let 2^ρ, 2^{ρ_k}, 2^{r_k}, 2^{α_k} be the highest powers of 2 dividing β, e_k, σ_k, w_{kk} respectively. Then the highest power of 2 which divides c_{kk} is 2^{γ_k}, where

$$\gamma_k = |\rho-\rho_k|+\alpha_k. \tag{8}$$

Thus $\mathfrak{C}$ is even if $|\rho-\rho_k|+\alpha_k \geqslant 1$ for $k=1, 2,\ldots, m$, and $\mathfrak{C}$ is odd if $\rho_k=\rho$ and $\alpha_k=0$ for some value of k.

Proof of Lemma 3. Writing $r=a/b$, we have

$$\begin{aligned} |G(\mathfrak{S}, r, \mathfrak{h})|^2 &= G(\mathfrak{S}, r, \mathfrak{h})G(\mathfrak{S}, -r, -h) \\ &= \sum_{\mathfrak{z}(b)}\sum_{\mathfrak{y}(b)} e^{2\pi i\{r(\mathfrak{z}'\mathfrak{S}\mathfrak{z}-\mathfrak{y}'\mathfrak{S}\mathfrak{y})+(\mathfrak{z}'-\mathfrak{y}')\mathfrak{h}\}} \\ &= \sum_{\mathfrak{x}(b)}\sum_{\mathfrak{y}(b)} e^{2\pi i\{r\mathfrak{x}'\mathfrak{S}\mathfrak{x}+2r\mathfrak{y}'\mathfrak{S}\mathfrak{x}+\mathfrak{x}'\mathfrak{h}\}}, \end{aligned}$$

* Siegel, **1**, Lemma 27 and (81).

on putting $\mathfrak{z} = \mathfrak{x}+\mathfrak{y}$. Now

$$\sum_{\mathfrak{y}(b)} e^{4\pi i r \mathfrak{y}'\mathfrak{S}\mathfrak{x}} = \begin{cases} b^m & \text{when } 2\mathfrak{S}\mathfrak{x} \equiv \mathfrak{n} \pmod{b}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$|G(\mathfrak{S}, r, \mathfrak{h})|^2 = b^m \sum_{\substack{\mathfrak{x}(b) \\ 2\mathfrak{S}\mathfrak{x}\equiv\mathfrak{n}(b)}} e^{2\pi i\{r\mathfrak{x}'\mathfrak{S}\mathfrak{x}+\mathfrak{x}'\mathfrak{h}\}}$$

$$\leqslant b^m \sum_{\substack{\mathfrak{x}(b) \\ 2\mathfrak{S}\mathfrak{x}\equiv\mathfrak{n}(b)}} 1 = b^m N(2\mathfrak{S}, b) \leqslant b^m 2^m S$$

by (7). We must consider

$$g(2\mathfrak{S}, r) = \sum_{\substack{\mathfrak{x}(b) \\ 2\mathfrak{S}\mathfrak{x}\equiv\mathfrak{n}(b)}} e^{2\pi i r \mathfrak{x}'\mathfrak{S}\mathfrak{x}} \tag{9}$$

more closely.

(i) Suppose b is odd. Then $2\mathfrak{S}\mathfrak{x} \equiv \mathfrak{n} \pmod{b}$ implies $\mathfrak{S}\mathfrak{x} \equiv \mathfrak{n} \pmod{b}$, so that, as $r\mathfrak{x}'\mathfrak{S}\mathfrak{x}$ is an integer for these values of $\mathfrak{x}$,

$$g(2\mathfrak{S}, r) = \sum_{\substack{\mathfrak{x}(b) \\ \mathfrak{S}\mathfrak{x}\equiv\mathfrak{n}(b)}} 1 = N(\mathfrak{S}, b).$$

(ii) Suppose b is even; say $b = 2\beta$. Then $2\mathfrak{S}\mathfrak{x} \equiv \mathfrak{n} \pmod{b}$ implies $\mathfrak{S}\mathfrak{x} \equiv \mathfrak{n} \pmod{\beta}$ so that, putting $\mathfrak{x} = \beta\mathfrak{y}+\mathfrak{z}$,

$$\begin{aligned} g(2\mathfrak{S}, r) &= \sum_{\substack{\mathfrak{x}(2\beta) \\ \mathfrak{S}\mathfrak{x}\equiv\mathfrak{n}(\beta)}} e^{2\pi i r\mathfrak{x}'\mathfrak{S}\mathfrak{x}} \\ &= \sum_{\substack{\mathfrak{z}(\beta) \\ \mathfrak{S}\mathfrak{z}\equiv\mathfrak{n}(\beta)}} \sum_{\mathfrak{y}(2)} e^{2\pi i r(\beta\mathfrak{y}+\mathfrak{z})'\mathfrak{S}(\beta\mathfrak{y}+\mathfrak{z})} \\ &= \sum_{\substack{\mathfrak{z}(\beta) \\ \mathfrak{S}\mathfrak{z}\equiv\mathfrak{n}(\beta)}} e^{2\pi i r\mathfrak{z}'\mathfrak{S}\mathfrak{z}} \sum_{\mathfrak{y}(2)} e^{\pi i a\beta\mathfrak{y}'\mathfrak{S}\mathfrak{y}} \\ &= \begin{cases} 0 & \text{if } \beta\mathfrak{S} \text{ is odd,} \\ 2^m g(\mathfrak{S}, a/\beta) & \text{if } \beta\mathfrak{S} \text{ is even,} \end{cases} \end{aligned}$$

since

$$\begin{aligned} \sum_{\mathfrak{y}(2)} e^{\pi i a\beta\mathfrak{y}'\mathfrak{S}\mathfrak{y}} &= \prod_{k=1}^{m} \left\{ \sum_{y_k=0}^{1} e^{\pi i a\beta s_{kk} y_k^2} \right\} \\ &= \begin{cases} 0 & \text{if some } \beta s_{kk} \text{ is odd,} \\ 2^m & \text{if } \beta s_{kk} \text{ is even for } k = 1, 2, \dots, m. \end{cases} \end{aligned}$$

(ii) (a) Suppose β is odd and $\mathfrak{S}$ is even. Then $(1/\beta)\mathfrak{x}'\mathfrak{S}\mathfrak{x}$ is an even integer for all integers $\mathfrak{x}$ such that $\mathfrak{S}\mathfrak{x} \equiv \mathfrak{n} \pmod{b}$. Hence

$$g(\mathfrak{S}, a/\beta) = \sum_{\substack{\mathfrak{x}(\beta) \\ \mathfrak{S}\mathfrak{x}\equiv\mathfrak{n}(\beta)}} 1 = N(\mathfrak{S}, \beta).$$

(ii) (b) Suppose β is even. Putting $\mathfrak{x} = \mathfrak{B}\mathfrak{y}$, as $\mathfrak{x}$ runs over a complete remainder system to modulus β so does $\mathfrak{y}$. Also

$$\mathfrak{S}\mathfrak{x} \equiv \mathfrak{n} \pmod{\beta}$$

is equivalent to
$$\mathfrak{D}\mathfrak{y} \equiv \mathfrak{n} \pmod{\beta},$$

i.e. to
$$e_k y_k \equiv 0 \pmod{\beta} \quad (k = 1, 2, ..., m).$$

The solutions of this system of congruences which are incongruent to modulus β are given by

$$y_k = (\beta/\sigma_k) z_k \quad (k = 1, 2, ..., m),$$

where z_k runs over a complete remainder system mod σ_k. With these transformations

$$\mathfrak{x}'\mathfrak{S}\mathfrak{x} = \mathfrak{y}'\mathfrak{B}'\mathfrak{S}\mathfrak{B}\mathfrak{y} = \mathfrak{y}'\mathfrak{W}\mathfrak{D}\mathfrak{y} = \beta\mathfrak{z}'\mathfrak{C}\mathfrak{z}.$$

Thus

$$g(\mathfrak{S}, a/\beta) = \sum_{z_1=1}^{\sigma_1} \cdots \sum_{z_m=1}^{\sigma_m} e^{\pi i a \mathfrak{z}'\mathfrak{C}\mathfrak{z}}$$
$$= \prod_{k=1}^{m} \left\{ \sum_{z_k=1}^{\sigma_k} e^{\pi i c_{kk} z_k} \right\},$$

since
$$a\mathfrak{z}'\mathfrak{C}\mathfrak{z} \equiv \sum_{k=1}^{m} c_{kk} z_k \pmod{2}.$$

Now, if c_{kk} is odd, then $\gamma_k = 0$ and $\rho_k = \rho$, so that σ_k contains the same power of 2 as β, i.e. σ_k is even. Thus

$$\sum_{z_k=1}^{\sigma_k} e^{\pi i c_{kk} z_k} = \begin{cases} 0 & \text{when } c_{kk} \text{ is odd,} \\ \sigma_k & \text{when } c_{kk} \text{ is even.} \end{cases}$$

Hence

$$g(\mathfrak{S}, a/\beta) = \begin{cases} 0 & \text{when } \mathfrak{C} \text{ is odd,} \\ \sigma_1 \ldots \sigma_m = N(\mathfrak{S}, \beta) & \text{when } \mathfrak{C} \text{ is even.} \end{cases}$$

This completes the proof of the lemma.

5. The function $f(\mathfrak{S}, \tau)$

We need the following inequalities for $|f(\tau)|$, where

$$f(\tau) = f(\mathfrak{S}, \tau) = \sum_{\mathfrak{x}} e^{\pi i \tau \mathfrak{x}'\mathfrak{S}\mathfrak{x}}.$$

They will serve instead of the modular transformation equation in the Hardy-Littlewood treatment.

LEMMA 4. *Suppose a, b are integers such that $(a, b) = 1$ $(b > 0)$. Let*

$$\omega = 2a - b\tau, \qquad \tau = x + iy, \qquad \lambda = \frac{y}{\{(2a-bx)^2 + b^2y^2\}}.$$

Then $$|f(\mathfrak{S},\tau)| \leqslant |2/\omega|^{m/2} f(\mathfrak{S}^{-1}, i\lambda) \quad (y>0), \tag{10}$$

and, when $G(\mathfrak{S}, a/b) \neq 0$,

$$|f(\mathfrak{S},\tau)| \geqslant \frac{1}{|\omega|^{m/2}}\left(\frac{N(2\mathfrak{S},b)}{S}\right)^{\frac{1}{2}}\Big\{1-\sum_{\mathfrak{x}}{}' e^{-\pi\lambda\mathfrak{x}'\mathfrak{S}^{-1}\mathfrak{x}}\Big\} \quad (y>0). \tag{11}$$

Proof. As all the series involved are absolutely convergent when $y>0$, we may rearrange the series by means of the substitution $\mathfrak{x} = b\mathfrak{y}+\mathfrak{z}$, and obtain

$$f(\mathfrak{S},\tau) = \sum e^{\pi i\tau\mathfrak{x}'\mathfrak{S}\mathfrak{x}} = \sum_{\mathfrak{y}}\sum_{\mathfrak{z}(b)} e^{\pi i\{(2a-\omega)/b\}(b\mathfrak{y}+\mathfrak{z})'\mathfrak{S}(b\mathfrak{y}+\mathfrak{z})}$$

$$= \sum_{\mathfrak{z}(b)} e^{2\pi i(a/b)\mathfrak{z}'\mathfrak{S}\mathfrak{z}} \sum_{\mathfrak{y}} e^{-\pi i(\omega/b)(b\mathfrak{y}+\mathfrak{z})'\mathfrak{S}(b\mathfrak{y}+\mathfrak{z})}$$

$$= \sum_{\mathfrak{z}(b)} e^{2\pi i(a/b)\mathfrak{z}'\mathfrak{S}\mathfrak{z}} \frac{S^{-\frac{1}{2}}}{(ib\omega)^{m/2}} \sum_{\mathfrak{y}} e^{\pi i\{(1/b\omega)\mathfrak{y}'\mathfrak{S}^{-1}\mathfrak{y}-(2/b)\mathfrak{y}'\mathfrak{z}\}}$$

by the reciprocal functional equation. That is,

$$f(\mathfrak{S},\tau) = \frac{S^{-\frac{1}{2}}}{(ib\omega)^{m/2}} \sum_{\mathfrak{y}} G(\mathfrak{S}, a/b, (-1/b)\mathfrak{y}) e^{\pi i(1/b\omega)\mathfrak{y}'\mathfrak{S}^{-1}\mathfrak{y}}. \tag{12}$$

Consequently, by Lemma 3,

$$|f(\mathfrak{S},\tau)| \leqslant |2/\omega|^{m/2} \sum_{\mathfrak{y}} e^{-\pi\lambda\mathfrak{y}'\mathfrak{S}^{-1}\mathfrak{y}} = |2/\omega|^{m/2} f(\mathfrak{S}^{-1}, i\lambda)$$

and

$$|f(\mathfrak{S},\tau)| \geqslant \frac{S^{-\frac{1}{2}}}{|b\omega|^{m/2}}\Big\{|G(\mathfrak{S},a/b)| - \{b^m N(2\mathfrak{S},b)\}^{\frac{1}{2}} \sum_{\mathfrak{y}}{}' e^{-\pi\lambda\mathfrak{y}'\mathfrak{S}^{-1}\mathfrak{y}}\Big\}$$

$$\geqslant \frac{1}{|\omega|^{m/2}}\left(\frac{N(2\mathfrak{S},b)}{S}\right)^{\frac{1}{2}}\Big\{1-\sum_{\mathfrak{y}}{}' e^{-\pi\lambda\mathfrak{y}'\mathfrak{S}^{-1}\mathfrak{y}}\Big\},$$

when $G(\mathfrak{S}, a/b) \neq 0$, since then by (5), (6), and Lemma 3

$$|G(\mathfrak{S},a/b)| = \{b^m N(2\mathfrak{S},b)\}^{\frac{1}{2}}.$$

6. Asymptotic behaviour of $f(\mathfrak{S},\tau)$

The behaviour of $f(\mathfrak{S},\tau)$ near the various classes of points on its natural boundary will now be discussed. First of all we have trivially

THEOREM 1. *If x is rational and equal to $2a/b$ where $(a,b)=1$, $b>0$, then*

$$f(\mathfrak{S},\tau) \sim \frac{G(\mathfrak{S},a/b)}{S^{\frac{1}{2}} b^m y^{m/2}}$$

as $y \to +0$.

Proof. In (12) of Lemma 4

$$\omega = 2a - b(x+iy) = -iby,$$

so that

$$f(\mathfrak{S},\tau) = \frac{S^{-\frac{1}{2}}}{(b^2y)^{m/2}} \sum_{\mathfrak{x}} G(\mathfrak{S}, a/b, (-1/b)\mathfrak{x}) e^{-(\pi/b^2y)\mathfrak{x}'\mathfrak{S}^{-1}\mathfrak{x}}$$

$$= \frac{G(\mathfrak{S}, a/b)}{S^{\frac{1}{2}} b^m y^{m/2}} + \Sigma_1,$$

where by Lemma 3

$$|\Sigma_1| \leqslant \left(\frac{2}{by}\right)^{m/2} \sum_{\mathfrak{x}}{}' e^{-(\pi/b^2y)\mathfrak{x}'\mathfrak{S}^{-1}\mathfrak{x}} = o\,(y^{-m/2}).$$

We shall suppose from now on that x is irrational. Let

$$\tfrac{1}{2}x = a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots$$

and p_r/q_r be the rth convergent to $\frac{1}{2}x$. Write a'_{r+1} for the complete quotient corresponding to a_{r+1} and

$$q'_{r+1} = a'_{r+1} q_r + q_{r-1},$$

so that

$$|\tfrac{1}{2}xq_r - p_r| = 1/q'_{r+1}.$$

We shall say that y lies in the range R_r if

$$2/q'^2_{r+1} \leqslant y \leqslant 2/q_r^2,$$

i.e., putting $y = 2\zeta/q_r q'_{r+1}$, $\eta = q'_{r+1}/q_r$,

$$1/\eta \leqslant \zeta \leqslant \eta.$$

Since

$$q'_{r+1} - q_{r+1} = (a'_{r+1} - a_{r+1})q_r > 0,$$

when x is irrational, we see that the range R_{r+1} overlaps R_r. Hence as y tends to zero it will always lie in at least one range R_r.

LEMMA 5. *If x is irrational and y lies in the range R_r, then*

$$|f(\mathfrak{S},\tau)| < K[\min(q'^2_{r+1}, 4/q_r^2 y^2)]^{m/4},$$

where K is a constant depending only on $\mathfrak{S}$.

Proof. Take $a = p_r$, $b = q_r$ in Lemma 4 so that

$$|\omega| = |2p_r - q_r x - iq_r y| = 2(1/q'^2_{r+1} + \tfrac{1}{4}q_r^2 y^2)^{\frac{1}{2}}$$

and

$$\lambda = \frac{y}{4/q'^2_{r+1} + q_r^2 y^2} = \frac{\zeta}{1+\zeta^2}\,\frac{q'_{r+1}}{2q_r}. \tag{13}$$

We notice that λ has its least value in R_r when

$$\zeta = \eta \quad \text{or} \quad 1/\eta$$

so that in R_r
$$\lambda \geqslant \frac{q_{r+1}'^2}{2(q_r^2+q_{r+1}'^2)} > \frac{1}{4}.$$
Hence
$$f(\mathfrak{S}^{-1}, i\lambda) = \sum_{\mathfrak{x}} e^{-\pi\lambda\mathfrak{x}'\mathfrak{S}^{-1}\mathfrak{x}}$$
$$< \sum_{\mathfrak{x}} e^{-\frac{1}{4}\pi\mathfrak{x}'\mathfrak{S}^{-1}\mathfrak{x}} = K.$$
Thus by Lemma 4
$$|f(\mathfrak{S},\tau)| < K(1/q_{r+1}'^2+\tfrac{1}{4}q_r^2 y^2)^{-m/4}$$
$$< K[\min(q_{r+1}'^2,\ 4/q_r^2 y^2)]^{m/4}$$
when y lies in R_r.

THEOREM 2. *For any irrational value of x*
$$f(\mathfrak{S}, x+iy) = o\,(y^{-m/2})$$
as $y \to 0$.

Proof. By Lemma 5
$$|f(\mathfrak{S}, x+iy)| < 2^{m/2}K(q_r y)^{-m/2} = o\,(y^{-m/2}),$$
since as $y \to 0$, $r \to \infty$ and $q_r \to \infty$.

THEOREM 3. *If $\frac{1}{2}x$ is an irrational number with bounded partial quotients, then*
$$f(\mathfrak{S}, x+iy) = O(y^{-m/4})$$
as $y \to 0$.

Proof. We have
$$q_{r+1}' = a_{r+1}' q_r + q_{r-1} < (a_{r+2}+2)q_r = O(q_r) = O(y^{-\frac{1}{2}}),$$
when y lies in R_r. Thus Lemma 5 gives
$$|f(\mathfrak{S}, x+iy)| < K q_{r+1}'^{m/2} = O(y^{-m/4}).$$

THEOREM 4. *Let $\phi(t)$ be any function which is positive, continuous, and monotonic increasing for large values of t and such that*
$$\int^{\infty} \frac{dt}{t\phi(t)}$$
is convergent. Then for almost all values of x*
$$f(\mathfrak{S}, x+iy) = O\left(\left[\frac{1}{y}\phi\left(\frac{1}{y}\right)\right]^{m/4}\right)$$
as $y \to 0$. In particular, for almost all values of x
$$f(\mathfrak{S}, x+iy) = O(y^{-m/4}\log^{\frac{1}{4}m+\epsilon}(1/y)).$$

* i.e. apart from a set of Lebesgue measure zero.

Proof. By Khintchine's lemma,* for almost all values of x

$$|\tfrac{1}{2}xq-p| > \frac{1}{q\phi(q)}, \qquad (p,q) = 1,$$

for all but a finite number of values of p and q. Hence,

$$1/q'_{r+1} = |\tfrac{1}{2}xq_r - p_r| > 1/q_r\,\phi(q_r)$$

if r is sufficiently large, say $r > r_0$. Thus, if y lies in R_r,

$$q'_{r+1}/q_r < \phi(q_r) < \phi\left(\frac{1}{y}\right) \qquad (r > r_0).$$

Substituting in Lemma 5 we obtain

$$|f(\mathfrak{S}, x+iy)| < K\left[\frac{2}{y}\phi\left(\frac{1}{y}\right)\right]^{m/4}.$$

THEOREM 5. *If x is an algebraic number of degree k*

$$f(\mathfrak{S}, x+iy) = O(y^{-\kappa})$$

as $y \to 0$, where $\kappa = m(k-1)/2k$.

Proof. As $\frac{1}{2}x$ is algebraic of degree k, we have by Liouville's theorem

$$1/q'_{r+1} = |\tfrac{1}{2}xq_r - p_r| > A/q_r^{k-1},$$

where A is a positive number depending on x only. We divide R_r into two parts.

(i) If $2/q_r\,q'_{r+1} \leqslant y \leqslant 2/q_r^2$, we have

$$y \geqslant 2/q_r\,q'_{r+1} > 2A/q_r^k,$$

so that $1/q_r = O(y^{1/k})$. Hence

$$|f(\mathfrak{S}, x+iy)| < 2^{m/2}K(q_r\,y)^{-m/2} = O(y^{-\kappa}).$$

(ii) If $2/q'^2_{r+1} \leqslant y \leqslant 2/q_r\,q'_{r+1}$, we have

$$q'^k_{r+1} = O\{(q_r\,q'_{r+1})^{k-1}\} = O(y^{1-k}).$$

Hence $$f(\mathfrak{S}, \tau) = O(q'^{m/2}_{r+1}) = O(y^{-\kappa}).$$

By (i) and (ii) the result holds throughout the range R_r and so generally.

7. Ω-theorems

Let μ be the smallest number such that

$$\sum_{\mathfrak{x}}{}' e^{-\pi t\mathfrak{x}'\mathfrak{S}^{-1}\mathfrak{x}} \leqslant \tfrac{1}{2},$$

when $t \geqslant \mu$. Then μ exists and depends on $\mathfrak{S}$ only. Thus in Lemma 4, if $G(\mathfrak{S}, a/b) \neq 0$ and $\lambda \geqslant \mu$,

$$|f(\mathfrak{S}, \tau)| \geqslant \tfrac{1}{2}S^{-\frac{1}{2}}|\omega|^{-m/2}. \tag{14}$$

* A. Khintchine, **1**.

In particular, if q_r is odd, we see from Lemma 3 that

$$G(\mathfrak{S}, p_r/q_r) \neq 0.$$

Now, if $a = p_r$, $b = q_r$, and $y = 2/q_r q'_{r+1}$,

$$\lambda = q'_{r+1}/4q_r, \qquad |\omega| = 2\sqrt{2}/q'_{r+1}.$$

Hence we have

LEMMA 6. *If $y = 2/q_r q'_{r+1}$, $q'_{r+1} > 4\mu q_r$ and q_r is odd, then*

$$|f(\mathfrak{S}, \tau)| > 2^{1-3m} S^{-\frac{1}{2}} q'^{m/2}_{r+1}. \tag{15}$$

THEOREM 6. *Let $\omega(t)$ be any positive continuous function which tends monotonically to zero as $t \to \infty$. Then corresponding to this function there are some irrational numbers x for which*

$$f(\mathfrak{S}, x+iy) = \Omega(y^{-m/2}\omega(1/y)).$$

Proof. It is enough to define a suitable sequence of partial quotients. First we notice that, since

$$p_{r+1} q_r - p_r q_{r+1} = (-1)^r,$$

it follows that q_{r+1} is odd when q_r is even. Having defined $a_1, a_2, \ldots, a_k$ in an arbitrary fashion and hence $p_1/q_1, \ldots, p_k/q_k$, we proceed to define the remaining partial quotients according to the following rule:

(i) If q_r is odd, choose a_{r+1} so large that

$$\omega(\tfrac{1}{2}a_{r+1} q_r^2) < q_r^{-m/2} \quad \text{and} \quad a_{r+1} > 4\mu.$$

(ii) If q_r is even, choose a_{r+1} arbitrarily, e.g. $a_{r+1} = 1$.

In the latter case q_{r+1} will be odd. Hence the number so defined is such that for an infinite sequence of values of r

$$q'_{r+1} > a_{r+1} q_r > 4\mu q_r,$$

$$q_r^{-m/2} > \omega(\tfrac{1}{2}a_{r+1} q_r^2) > \omega(\tfrac{1}{2}q_r q'_{r+1}) = \omega\left(\frac{1}{y}\right)$$

and q_r is odd. Hence by Lemma 6

$$\begin{aligned} |f(\mathfrak{S}, \tau)| &> 2^{1-3m} S^{-\frac{1}{2}} q'^{m/2}_{r+1} \\ &= 2^{1-5m/2} S^{-\frac{1}{2}} y^{-m/2} q_r^{-m/2} \\ &> 2^{1-5m/2} S^{-\frac{1}{2}} y^{-m/2} \omega\left(\frac{1}{y}\right) \end{aligned}$$

as $y \to 0$ through the sequence of values corresponding to the sequence of values of r. The result follows.

We next require a slightly modified form of Khintchine's lemma.

LEMMA 7. *Suppose that $\psi(t)$ is a positive, continuous, monotonically increasing function such that*

$$\int^{\infty} \frac{dt}{t\psi(t)}$$

is divergent. Then for almost all values of θ

$$|\theta q - p| < \frac{1}{q\psi(q)}$$

has an infinite number of integer solutions p, q such that

$$(p,q) = 1, \qquad q > 0 \quad \textit{and} \quad q \equiv 1 \pmod 2).$$

The only difference between this and Khintchine's lemma lies in the extra assertion that q is odd in an infinite number of the solutions. It is easy to adapt Khintchine's later metrical proof* to justify this addition.

THEOREM 7. *Suppose that $\psi(t)$ fulfils the conditions of Lemma 7 and that*

$$\psi(t^2) = O\{\psi(t)\}$$

as $t \to \infty$. Then for almost all values of x

$$f(\mathfrak{S}, x+iy) = \Omega\left(\left[\frac{1}{y}\psi(1/y)\right]^{m/4}\right).$$

In particular, for almost all values of x

$$f(\mathfrak{S}, x+iy) = \Omega\{y^{-m/4}\log^{m/4}(1/y)\}.$$

Proof. By Lemma 7 for almost all x

$$|\tfrac{1}{2}xq - p| < 1/q\psi(q)$$

has an infinity of solutions such that q is odd and $(p,q) = 1$. But, if

$$|\tfrac{1}{2}xq - p| < 1/2q,$$

then† p/q must be a convergent of $\frac{1}{2}x$. Hence for almost all x there exists an infinity of convergents with odd denominator such that

$$1/q'_{r+1} = |\tfrac{1}{2}xq_r - p_r| < 1/q_r\psi(q_r).$$

Let Q denote this sequence of values of q_r. For all sufficiently large q_r in Q we have

$$\psi(q_r) > 4\mu.$$

Hence, if $y = 2/q_r q'_{r+1}$ and q_r is sufficiently large in Q, Lemma 6 gives‡

$$|f(\mathfrak{S}, \tau)| > Cq'^{m/2}_{r+1} > Cy^{-m/4}\{\psi(q_r)\}^{m/4}. \tag{16}$$

* A. Khintchine, **2**.

† See, for instance, O. Perron, **1**, 45.

‡ The constant C is not the same number in each occurrence, but it is always independent of q_r and y.

Now* for almost all x $\qquad q'_{r+1} = O(q_r^2)$.
Hence for this set of numbers x

$$\psi(1/y) < \psi(q_r q'_{r+1}) < \psi(Cq_r^3) < C\psi(q_r), \tag{17}$$

provided that r is large enough. Combining (16) and (17) we obtain the desired result.

8. A lattice-point problem

Let
$$R(N,x) = R(\mathfrak{S},N,x) = \sum_{Q(\mathfrak{y})\leqslant N} e^{\pi i x Q(\mathfrak{y})}$$
$$= \sum_{n=0}^{N} A(\mathfrak{S},n)e^{\pi i x n},$$

where $A(\mathfrak{S},n)$ is the number of integer columns $\mathfrak{y}$ such that

$$Q(\mathfrak{y}) = n.$$

The Ω-theorems proved for $f(\mathfrak{S},\tau)$ easily lead to analogues for $R(\mathfrak{S},N,x)$.

Theorem 8. *Let $\omega(t)$ be any positive continuous function which tends monotonically to zero as $t\to\infty$. Then there exist corresponding irrational numbers x for which*

$$R(\mathfrak{S},N,x) = \Omega\{N^{m/2}\omega(N)\}.$$

Proof. Suppose the theorem false. Then for any irrational number x

$$R(\mathfrak{S},N,x) = o\{N^{m/2}\omega(N)\}.$$

Thus for any irrational x

$$f(\mathfrak{S},\tau) = \sum_{n=0}^{\infty} A(\mathfrak{S},n)e^{\pi i x n-\pi y n}$$
$$= (1-e^{-\pi y})\sum_{n=0}^{\infty} R(n,x)e^{-\pi y n}$$
$$= o\Big(y\sum_{n=0}^{\infty} n^{m/2}\omega(n)e^{-\pi y n}\Big).$$

We may suppose without loss of generality that

$$\omega(t) > 1/t \quad \text{as} \quad t\to\infty.$$

Hence, putting $Y = [y^{-\frac{1}{2}}]$,

$$f(\mathfrak{S},\tau) = o\Big(y\sum_{n=0}^{Y} n^{m/2}\Big) + o\Big(y\omega(Y)\sum_{n=Y+1}^{\infty} n^{m/2}e^{-\pi y n}\Big)$$
$$= o(y^{\frac{1}{2}-m/4}) + o(y^{-m/2}\omega(1/\sqrt{y}))$$
$$= o(y^{-m/2}\omega(1/\sqrt{y}))$$

for any irrational x. But this contradicts Theorem 6, hence the result holds.

* See Khintchine, 1.

THEOREM 9. *Suppose that $\psi(t)$ is a positive, continuous, monotonically increasing function such that*

$$\int^{\infty} \frac{dt}{t\psi(t)}$$

is divergent and $$\psi(t^2) = O\{\psi(t)\}$$

as $t \to \infty$. Then for almost all x

$$R(\mathfrak{S}, N, x) = \Omega([N\psi(N)]^{m/4}).$$

In particular, for almost all x

$$R(\mathfrak{S}, N, x) = \Omega(N^{m/4}\log^{m/4}N).$$

Proof. Suppose the theorem false. Then for an x-set of positive outer Lebesgue measure

$$R(N, x) = o([N\psi(N)]^{m/4}).$$

Hence for an x-set of positive outer Lebesgue measure

$$\begin{aligned} f(\tau) &= (1-e^{-\pi y}) \sum_{n=0}^{\infty} R(n, x)e^{-\pi y n} \\ &= o\Big(y \sum_{n=0}^{\infty} [n\psi(n)]^{m/4}e^{-\pi y n}\Big) \\ &= o\Big(y \int_0^{\infty} [t\psi(t)]^{m/4}e^{-\pi y t}\, dt\Big) \\ &= o\Big(y^{-m/4} \int_0^{\infty} [u\psi(u/y)]^{m/4}e^{-\pi u}\, du\Big) \end{aligned}$$

Now, as we may suppose $\psi(t) > 1$ for $t > 0$,

$$\begin{aligned} \psi(u/y) \leqslant \max\{\psi(u^2), \psi(1/y^2)\} &< \psi(u^2)\psi(1/y^2) \\ &= O\{\psi(u)\psi(1/y)\}. \end{aligned}$$

Hence for an x-set of positive outer Lebesgue measure

$$f(\tau) = o(y^{-m/4}\{\psi(1/y)\}^{m/4}).$$

But this contradicts Theorem 7, hence the desired result is true.

We conclude by deducing

THEOREM 10. *Let $\phi(t)$ be any positive, continuous, monotonic increasing function such that*

$$\int^{\infty} \frac{dt}{t\phi(t)}$$

is convergent. Then

$$\sum_{n=0}^{\infty} \{A(\mathfrak{S}, n)\}^2 e^{-2\pi y n} = O\Big(\Big[\frac{1}{y}\phi(1/y)\Big]^{m/2}\Big).$$

In particular, for any $\epsilon > 0$,

$$\sum_{n=0}^{\infty} \{A(\mathfrak{S}, n)\}^2 e^{-2\pi y n} = O\{y^{-m/2} \log^{\frac{1}{2}m+\epsilon}(1/y)\}.$$

Proof. We have

$$\sum_{n=0}^{\infty} \{A(\mathfrak{S}, n)\}^2 e^{-2\pi y n} = \tfrac{1}{2} \int_{-1}^{1} |f(\tau)|^2 \, dx$$
$$= O([(1/y)\phi(1/y)]^{m/2})$$

by Theorem 4.

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AN APPLICATION OF FOURIER TRANSFORMS TO THE EVALUATION OF A CLASS OF REPEATED INTEGRALS ARISING IN CALCULATIONS OF PROBABILITY

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1. A CLASS of repeated integrals occurring in some physical problems is of the form

$$\int \cdots \int f_n(t_n) f_{n-1}(t_{n-1}) \cdots f_1(t_1)\, dt_1\, dt_2 \ldots dt_n,$$

where the variables $t_1, t_2, \ldots, t_n$ are restricted by the condition

$$s \leqslant t_1 + t_2 + \ldots + t_n \leqslant s + ds.$$

This can be reduced immediately to the form

$$ds \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} f_n(s - y_{n-1}) f_{n-1}(y_{n-1} - y_{n-2}) \cdots f_2(y_2 - y_1) f_1(y_1)\, dy_1 \ldots dy_{n-1},$$

i.e. an n-tuple integral is reduced to an $(n-1)$tuple integral and the ranges determined.

As an example of the origin of these forms, let $f(x)\, dx$ be the probability of a displacement between x and $x+dx$, due, say, to Brownian movement in a certain interval, then, taking

$$f_1 = f_2 = \ldots = f_n = f,$$

$$ds \int \cdots \int f(x_n) f(x_{n-1}) \cdots f(x_1)\, dx_1 \ldots dx_n,$$

where $s \leqslant \sum x_r \leqslant s + ds$, is $\phi(s)\, ds$, the probability of a displacement between s and $s+ds$ occurring as the resultant of n successive displacements under like circumstances.

Transforming, we have to evaluate

$$\int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} f(s - y_{n-1}) f(y_{n-1} - y_{n-2}) \cdots f_2(y_2 - y_1) f_1(y_1)\, dy_1 \ldots dy_{n-1}.$$

Analytically this expression is unwieldy and its properties, as a function of s, can only be extracted with difficulty. The numerical evaluation of $\phi(s)$ when n is more than 2 or 3 is clearly a very laborious task.

It is the object of this note to show that this problem is capable of much simpler and more elegant analytical formulation, and

further that thereby the numerical computations referred to are greatly simplified even for a rigorous solution, and can be reduced to a very simple general approximate form applicable in nearly all practical cases.

2. Let the functions $f_r(t)$ be bounded along the real axis, positive at all points of it, and tending to zero at infinity in both directions so that for each f_r the Fourier transform g_r exists and has the usual properties so that

$$g(u) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(t)e^{-iut}\,dt,$$

$$f(t) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} g(u)e^{iut}\,du.$$

If f be not continuous over the whole range, the integrals are to exist as Lebesgue integrals. Further, let the functions be normalized so that

$$\int_{-\infty}^{\infty} f(t)\,dt = 1,$$

and the origin so chosen that

$$\int_{-\infty}^{\infty} tf(t)\,dt = 0.$$

We now proceed to build up step by step the repeated integrals to be evaluated and show that the Fourier transform at each stage is simply the product of the transforms of the individual functions.

Let $\quad F_1(s) = f_1(s),$

$$F_2(s)\,ds = \int\int f_2(x)f_1(x)\,dx_1\,dx_2,$$

. . . .

$$F_n(s)\,ds = \int \ldots \int f_n(x_n)\ldots f_1(x_1)\,dx_1\ldots dx_n,$$

where $s \leqslant x_1+x_2+\ldots+x_r \leqslant s+ds$, for $F_r(s)$. Now consider F_2, where $s \leqslant x_1+x_2 \leqslant s+ds$, and put $x_1 = t$, $x_2 = s-t$, then

$$F_2(s) = \int_{-\infty}^{\infty} f_2(s-t)f_1(t)\,dt,$$

and it is readily seen that

$$F_r(s) = \int_{-\infty}^{\infty} f_r(s-t)\,F_{r-1}(t)\,dt.$$

Let
$$G_r(u) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F_r(t)e^{-iut}\,dt$$
$$= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{-iut} \int_{-\infty}^{\infty} f_r(t-v)F_{r-1}(v)\,dvdt.$$

By Fubini's theorem we may change the order of integration and obtain
$$G_r(u) = (2\pi)^{\frac{1}{2}}g_r(u)\,G_{r-1}(u).$$

Thus
$$G_n(u) = (2\pi)^{\frac{1}{2}(n-1)}g_n(u)\,g_{n-1}(u)\ldots g_1(u) \tag{I}$$

and
$$F_n(s) = (2\pi)^{\frac{1}{2}n-1} \int_{-\infty}^{\infty} e^{ius}g_n(u)\,g_{n-1}(u)\ldots g_1(u)\,du, \tag{II}$$

where $F_n(s)$ is the required 'distribution' function.

As it sometimes happens that the g's are known (or easily found), this formula is itself a useful aid, and simplifies numerical work. However, we proceed to a still greater simplification.

3. An asymptotic expression for $F_n(s)$.

The method of steepest descents leads at once to the result that for large n the form of $F_n(s)$ is $e^{-a^2s^2}$ and determines a in terms of the second moments of the individual distribution functions f.

We have
$$\frac{d}{du}(g_1 g_2 \cdots g_n) = g_1 g_2 \cdots g_n \sum \frac{g_r'}{g_r},$$

and the least zero is at $u = 0$, where there is in general a maximum maximorum of $G(u)$. All g's possess this same property and $g(0) = 1/\sqrt{(2\pi)}$, in virtue of the normalizing conditions.

Thus near $u = 0$
$$g(u) = \frac{1}{\sqrt{(2\pi)}}\left\{1 - \frac{u^2}{2}\mu_2 - i\frac{u^3}{6}\mu_3 + \frac{u^4}{24}\mu_4 \cdots\right\},$$

where
$$\mu_r = \int_{-\infty}^{\infty} t^r f(t)\,dt.$$

Thus
$$g(u) = e^{\log g(u)} = \frac{1}{\sqrt{(2\pi)}} e^{-\frac{1}{2}\mu_2 u^2}\left\{1 + \frac{\mu_4 - 3\mu_2^2}{24}u^4 + O(u^5)\right\}$$

(omitting the term in u^3 which disappears on subsequent integration).

Thus
$$G_n(u) = \frac{1}{\sqrt{(2\pi)^n}} e^{-\frac{1}{2}Mu^2}\left\{1 + \left(nu^4\frac{(\mu_4 - 3\mu_2^2)}{24}\right) + O(u^5)\right\},$$

where $M = \sum_{p=1}^{n} {}_p\mu_2$, and diverges to ∞ with n, and

$${}_p\mu_2 = \int_{-\infty}^{\infty} t^2 f_p(t)\, dt.$$

Thus $$F_n(s) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ius-\frac{1}{2}Mu^2}\, du \sim \frac{1}{\sqrt{(2\pi M)}} e^{-\frac{1}{2}(s^2/M)}. \tag{III}$$

This satisfies the condition of consistency, i.e. that $\int_{-\infty}^{\infty} F(s)\, ds = 1$, and is the required result.

In the commonly occurring special case where

$$f_n = f_{n-1} = \dots = f_1 = f \text{ say,}$$
$$M = n\mu_2,$$

and $$F_n(s) \sim \frac{1}{\sqrt{(2\pi\mu_2 n)}} e^{-\frac{1}{2}(s^2/n\mu_2)}. \tag{IV}$$

The order of the omitted terms will be

$$\left\{\frac{1}{8n} \frac{(\mu_4 - 3\mu_2^2)}{\mu_2^2}\right\}$$

of the leading term, and this tends to zero as n tends to ∞.

Incidentally it may be remarked that this constitutes a proof that the resultant of a large number n of sources of errors of observation is distributed according to the 'normal error law', however the individual sources may be distributed. This is a well-known result; this note adds to it by showing that the 'convergence' is rapid, and allowing the parameter of the distribution function to be evaluated and the order of approximation to be calculated, for relatively small n.

4. An actuarial problem

In practice these results are much more convenient than is at first apparent. For convenience of reference by those wishing to use the results without disentangling them from the proof, I cite the example of an actuarial problem.

A typical question, of practical importance, requiring too elaborate computation for answering in the ordinary course is as follows. A set of individuals contract to pay an annual premium during life, from entry, to a pension scheme until a fixed age, say 65, and on survival to that age enjoy a simple life annuity of appropriate amount. The

ordinary computations of the 'net' premiums for an annuity (that is, excluding the 'loading' of the premium by adding an amount deemed sufficient by the insurers to cover their operating costs and provide a suitable profit) are based on life tables and are such that the expectation of either a profit or a loss is just one-half. It is clearly important to assess the probability that the profit or loss on a group of n individuals will exceed some specified figure. This problem is essentially of the type dealt with above.

For simplicity of illustration we will consider a group of n individuals, all entering simultaneously at age 35 on a scheme to provide at 65 a deferred annuity of amount A per annum for a premium (not returnable) of p during the 30 years' interval of deferment. If δ_x represents the probability of dying at age x, and v the present value of £1 due in one year at the rate of interest used in calculating p $\{= AN_{x+n}/(N_x - N_{x+n})\}$, then considering a single individual we have, reckoning in terms of present values at the inception of the scheme,

δ_x is the probability of a profit p

$$\begin{array}{llll}
\delta_{x+1} & ,, & ,, & ,, \quad p(1+v) \\
\delta_{x+n} & ,, & ,, & ,, \quad p(1+v+\ldots+v^n) = p\dfrac{1-v^{n+1}}{1-v} \\
\delta_{x+n+1} & ,, & ,, & ,, \quad p\dfrac{1-v^{n+1}}{1-v} - Av^{n+1} \\
\delta_{x+n+r} & ,, & ,, & ,, \quad p\dfrac{1-v^{n+1}}{1-v} - Av^{n+1}\dfrac{1-v^r}{1-v}.
\end{array}$$

We have our usual conditions

$$\sum \delta_x = 1, \qquad \sum (\text{profit})\delta = 0.$$

We require μ_2 the second moment

$$= \sum (\text{profit})^2\delta.$$

There will as a rule be two ways of gaining a given profit P (either by death before maturity or death soon after maturity), but it is not necessary to find the resultant of these as in $\sum (\text{profit})^2\delta$ the relevant term will be $P^2(\delta_1+\delta_2)$, i.e. we need only form the second moment μ_2 of the table as it stands, a simple computation. The required information is then given by

$$P(s) = \frac{1}{\sqrt{(2\pi\mu_2 n)}} e^{-\frac{1}{2}(s^2/n\mu_2)},$$

where $P(s)\,ds$ is the probability of a profit (or loss) between s and $s+ds$, and the likelihood of a loss exceeding L is

$$\frac{1}{\sqrt{(2\pi\mu_2 n)}}\int\limits_L^\infty e^{-(s^2/2\mu_2 n)}\,ds,$$

and reference to a table of error integrals gives the result immediately.

To quote a numerical example, based on life tables and commutation columns at $3\frac{1}{2}$ per cent., we find:

For an annuity of £800 per annum the net premium is £95 per annum, and

$$\mu_2 = 3{\cdot}2677, \qquad \mu_4 = 21{\cdot}6144,$$

taking of course $A = 1$ in normalizing the distribution function. Thus the probability of a profit or loss s to $s+ds$ is

$$\frac{1}{\sqrt{(6{\cdot}5354\pi n)}}e^{-(s^2/6{\cdot}5354n)}\,ds,$$

the neglected term being $1/8n$ of the leading term; thus for $n = 30$ the error is 0·4 per cent., for $n = 3{,}000$ it is 0·004 per cent. The table below shows the use of the expression obtained.

If all the members are not of the same age and the time chosen is not at the inception of the scheme, the calculations are only modified in the relatively trivial respect that μ_2 must be found for each member, and M, the sum of these, used instead of n times μ_2.

Balance Sheet for Deferred Annuity Scheme at inception, with n members all at age 35 *contracting for an annuity of* £800 *p.a., at* 65

Number of members	*Mean present value of future premiums or annuities payable*	*Present value of loss sustainable will exceed figures below once in* 4 *times* (*a*)	10 *times* (*b*)	*Net premium*	*Additional premium required to cover risk of loss* (*a*)	(*b*)
	£	£	£	£	£	£
30	47,530	5,344	10,153	95	10·7	20·3
120	190,120	10,688	20,306	95	5·3	10·2
3,000	4,753,000	53,440	101,530	95	1·1	2·0
12,000	19,012,000	106,880	203,010	95	0·5	0·1

The risk contemplated here is that of an unusual proportion of the annuitants surviving to a great age and so throwing a large burden on the scheme. The actual sum lost can be computed, but for simplicity it is here stated as its present value at the inception of the scheme. The corresponding value at the start of payment of annuities is approximately 2·8 times these figures and for the time (60 years later) when all members can be supposed dead 2·8 times this again. The importance of a large number of participants is clearly seen.

THE MINIMUM MODULUS OF INTEGRAL FUNCTIONS OF FINITE ORDER

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IN Pólya's theory of gap theorems an important part is played by the following

THEOREM. *If $f(z)$ is an integral function satisfying* $\log M(r) = o(r)$, *where $M(r)$ is the maximum modulus of $f(z)$ for $|z| = r$, and $\alpha > 0$, then $|f(n)| > e^{-\alpha n}$ for almost all integers n.**

As many difficult proofs of this theorem and its generalizations have been published, it may be worth showing that these results are trivial corollaries of Hadamard's minimum-modulus theorem, at least when the precise form to be derived from a well-known theorem of Cartan† is used. Cartan's theorem states: *if H is any positive number, the polynomial $P(z) = (z-a_1)(z-a_2)...(z-a_n)$ satisfies $|P(z)| > (H/e)^n$ for z outside a system of at most n circles the sum of whose radii is at most $2H$.* We also require Nevanlinna's formula‡

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| \frac{R^2-r^2}{R^2+r^2-2rR\cos(\theta-\phi)}\, d\theta + $$
$$+ \sum_{|a_n|<R} \log\left|\frac{R(z-a_n)}{R^2-z\bar{a}_n}\right| \quad (R > r;\ z = re^{i\phi})$$

* The theorem was stated by Faber, *Jahresb. der Deutschen Math. Ver.* 16 (1907), 285–98. A proof was indicated by Pólya, *Comptes rendus*, 184 (1927), 579–81. For generalizations and further references see Pennycuick, *J. of London Math. Soc.* 12 (1937), 267–72. The applications to gap theorems are developed by Pólya, *Annals of Math.* (2) 34 (1933), 731–77, in particular 745–54.

† H. Cartan, *Ann. Sc. de l' École Norm.* (3) 45 (1928), 255–346. The theorem quoted is proved in pp. 272–6. There is an omission in the proof which is rectified in the version given by Valiron, *Directions de Borel des fonctions méromorphes*, Paris (1938), 11–12. For Theorem 2 we only need a much earlier result due to Boutroux, *Acta Math.* 28 (1904), 97–224 (129–33) and Theorem 2 itself is practically equivalent to Theorem 27, p. 89 of Valiron's *Lectures on the general theory of integral functions* (1923). A. Bloch, *Ann. Sc. de l' École Norm.* (3) 43 (1926), 309–62, states without proof a theorem very similar to that of Cartan. Both Bloch and Cartan proceeded to other matters without explicitly stating the inequality (1), which is in effect the conjunction of Bloch's lemmas 4 and 5 (pp. 320–3). Formal theorems on the minimum modulus of a function regular in a circle arising out of Cartan's theorem have been given by H. Milloux, *Proc. Phys. Math. Soc. Japan* (3) 12 (1930), and V. Bernstein, *Annali di Mat.* (4) 12 (1934), 179–81. (For the last two references I have to thank a referee.)

‡ R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Paris (1929), p. 24.

where $f(z)$ is supposed an integral function and a_n are its zeros. Evidently

$$\log|f(z)| \geqslant \frac{R-r}{R+r}m(R,f)-\frac{R+r}{R-r}m(R,1/f)+\sum_{|a_n|<R}\log\left|\frac{z-a_n}{2R}\right|,$$

where
$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta.$$

From Cartan's theorem with $H = hR$ we obtain

$$\log|f(z)| \geqslant \frac{R-r}{R+r}m(R,f)-\frac{R+r}{R-r}m(R,1/f)-n(R)\log\frac{e}{2h} \tag{1}$$

outside a set of at most $n(R)$ circles the sum of whose radii is at most $2hR$, where $n(R)$ denotes the number of zeros of $f(z)$ of modulus less than R). If h is small, it is evident that this inequality is satisfied for the majority of values of z, but there is some interest in determining the largest regions of various types in which the inequality holds throughout. The natural regions to consider are circles and annuli of the form $a < |z| < b$. Let us draw corresponding to each circle of radius $h_k R$ in which (1) is not true a concentric circle of radius $h_k R+\delta Rn(R)^{-\frac{1}{2}}$. The area included in this family of circles is at most

$$\pi R^2 \sum \{h_k+\delta n(R)^{-\frac{1}{2}}\}^2 \leqslant \pi R^2\{4h^2+4h\delta n(R)^{-\frac{1}{2}}+\delta^2\}.$$

Points outside these circles are the centres of circles of radii $\delta Rn(R)^{-\frac{1}{2}}$ throughout which (inside $|z| = R$) (1) holds. If h and δ are small, this applies to the majority of values of z. In order to obtain a more simple result let us suppose that $f(z)$ is of minimum type with respect to a proximate order $\rho(r)$, that is

$$\log M(r) = o\{V(r)\}, \qquad V(r) = r^{\rho(r)},$$

$$\lim_{r\to\infty}\frac{V(kr)}{V(r)} = k^\rho \quad (0 < k < \infty),$$

and let us confine our attention to values of r between $\frac{1}{4}R$ and $\frac{1}{2}R$. Then, if h, δ, and α are positive, no matter how small, (1) gives

$$\log|f(z)| > -\alpha V(r),$$

provided that r (and hence R) is sufficiently large. Also the radius $\delta Rn(R)^{-\frac{1}{2}} > rV(r)^{-\frac{1}{2}}$ for sufficiently large r, and we have

THEOREM 1. *If $f(z)$ is of minimum type with respect to a proximate order $\rho(r)$, then for any positive α*

$$\log|f(z)| > -\alpha V(r)$$

throughout the circle $|z-\zeta| < rV(r)^{-\frac{1}{2}}$ for almost all ζ.

Similarly, if we associate with a circle in which (1) does not hold, of radius h_k and centre distant r_k from the origin, the annulus

$$r_k - h_k R - \delta R n(R)^{-1} < |z| < r_k + h_k R + \delta R n(R)^{-1},$$

the total measure of such values of $|z|$ is not greater than $(4h+2\delta)R$. Arguing as before, we have

THEOREM 2. *If $f(z)$ is of minimum type with respect to a proximate order $\rho(r)$, then for every positive α*

$$\log|f(z)| > -\alpha V(r)$$

*throughout the annulus $r - r/V(r) < |z| < r + r/V(r)$ for almost all r.** It is evident that this result includes the result of Pólya by taking $\rho(r) = 1$.

* In the special case $V(r) = r^{\rho}$ these two theorems become the Theorems 4 and 5 given by Pennycuick, loc. cit.

The method extends to functions regular in an angle by using the corresponding form of Nevanlinna's formula. See Nevanlinna, loc. cit. 1–3 and *Acta Soc. Sc. Fennicae*, 50, No. 12 (1925), 29. It may be that referred to by Pólya, *Annals of Math.* loc. cit. 745 (in the footnote).

ON THE OCCURRENCE OF MILNE'S SYSTEMS OF PARTICLES IN GENERAL RELATIVITY

C. GILBERT (*Dundee*)

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1. THE systems of particles constructed by Milne* are of two types. These are (*a*) the simple kinematic system comprising a set of fundamental particles having a hydrodynamic motion, (*b*) the statistical systems defined by certain space-velocity distribution functions. A universe consists of the system (*a*) and a system (*b*). The system (*a*) is then a representation of a universe in which the mass of all the particles of the statistical component vanishes, and has therefore been called the substratum or smoothed-out universe. The types of equivalence which these particles possess have been clearly stated by Milne,† and I only wish to recall that both systems are described by a set of observers moving with the fundamental particles of the system (*a*), and that in the experience of every one of these observers the same picture of world events is obtained.

The first step in attempting to describe these systems according to the theory of general relativity is to determine the metric of space-time for the universe, and, since time and distance measurements have a significance only when they are taken in relation to the fundamental observers, the metric must be one corresponding to the system (*a*). The most general form of metric which, according to general relativity, corresponds to this system, has been shown by Walker‡ to be

$$ds^2 = \{F(X)\}^2\left\{dt^2 - \frac{1}{c^2}(dx^2+dy^2+dz^2)\right\}. \tag{1}$$

In this space-time the light tracks are given by the null geodesics, and the time and distance measurements are made according to Milne's light-signalling conventions from observations of clocks carried by the fundamental observers. The form of $F(X)$ can be further determined from the condition that the geodesic equations shall correspond to the equations of motion of certain of Milne's particles.

* Cf. *Relativity, Gravitation, and World Structure* (1935). Subsequently referred to as *W.S.* When other references to papers by E. A. Milne or A. G. Walker recur, they will be signified by name and date.

† *Proc. Royal Soc.* A, **156** (1936), 62.

‡ *M.N.* **95** (1935), 263.

2. According to Milne's theory the acceleration of a particle of the universe, having the coordinates and velocity $\mathbf{P}$, t, $\mathbf{V}$, in the experience of a fundamental observer, is*

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{\mathbf{V}}{Y^{\frac{1}{2}}}\right) = \frac{G(\xi)}{X}\left(\mathbf{P}-\frac{\mathbf{V}Z}{Y}\right), \tag{2}$$

where

$$X = t^2-\frac{\mathbf{P}^2}{c^2}, \qquad Y = 1-\frac{\mathbf{V}^2}{c^2}, \qquad Z = t-\frac{\mathbf{P}.\mathbf{V}}{c^2}, \qquad \xi = \frac{Z^2}{XY}. \tag{3}$$

The particle may be a free test-particle projected into space in any manner from one of the fundamental particles, and then Milne has shown† that, in the presence of the substratum alone, the acceleration function $G(\xi)$ has the value

$$G(\xi) \equiv -1. \tag{4}$$

On the other hand, if the particle be a member of a statistical system having the space-velocity distribution

$$\frac{\psi(\xi)}{c^6 X^{\frac{3}{2}} Y^{\frac{5}{2}}} dxdydzdudvdw, \tag{5}$$

the acceleration function takes the form

$$G(\xi) = -1-\frac{C}{(\xi-1)^{\frac{3}{2}}\psi(\xi)}, \tag{6}$$

where C is a constant, which has been shown‡ to be a measure of the gravitational mass of the particles.

3. In a space-time having the metric (1) the geodesic equations can be written in the form (2), if instead of $G(\xi)$ we write $G(X)$, where

$$G(X) = \frac{2XF'(X)}{F(X)}. \tag{7}$$

There is therefore a correspondence between the trajectories of particles in Milne's theory and the geodesic paths of general relativity when the acceleration function has a constant value.§ For example, the trajectories having $G(\xi) = -2n$ correspond to the geodesic paths in a space-time having the metric (1) with

$$F(X) = \left(\frac{t_0^2}{X}\right)^n. \tag{8}$$

The constant t_0, having the dimensions of time, has been introduced

* Milne (1936).

† *Quart. J. of Math.* (Oxford), **8** (1937), 22.

‡ Milne (1936).

§ Walker (1935).

into (8) to give s the dimensions of time. If we take $n = \frac{1}{2}$, the metric has the form

$$d\tau^2 = \frac{t_0^2}{X}\left\{dt^2 - \frac{1}{c^2}(dx^2+dy^2+dz^2)\right\}, \tag{9}$$

and the acceleration function has the value (4) for a free particle in the substratum. We shall subsequently show that this is the metric of space-time in which a description of Milne's systems of particles can be made. We therefore call (9) the *metric of the substratum.*

The following work is concerned with the construction of statistical systems of particles, when the observer's space-time has these possible forms, and is based on a recent paper by Walker* in which he derives the Boltzmann equations in general relativity.

4. The metric given by (1) and (8) is conformal to the metric of special relativity defined by the fundamental tensor g_{ij} having Galilean values. The fundamental tensors of the two spaces will be in the relation

$$\bar{g}_{ij} = e^{2\sigma}g_{ij}, \qquad \sigma = -n\log\left(\frac{X}{t_0^2}\right). \tag{10}$$

For convenience we shall often use a suffix notation, the suffixes taking the values 0 and 1, 2, 3 for the time-like and space-like components of a vector. Then

$$X = g_{hk}x^h x^k, \qquad Y = g_{hk}\frac{dx^h}{dt}\frac{dx^k}{dt}, \qquad Z = g_{hk}x^h\frac{dx^k}{dt} \tag{11}$$

and the metric (1) becomes

$$ds^2 = \bar{g}_{ij}\,dx^i dx^j. \tag{12}$$

Hence, if λ^i is the unit vector tangent to the path of a particle at an event P, we have

$$\bar{g}_{ij}\lambda^i\lambda^j = 1. \tag{13}$$

Then the element of three-dimensional solid angle $d\Omega$, determined by the range of directions $(\lambda^i, \lambda^i+d\lambda^i)$ relative to an observer at P, is given by†

$$d\Omega = (-\bar{g})^{\frac{1}{2}}(\lambda_0)^{-1}\,d\lambda^1 d\lambda^2 d\lambda^3. \tag{14}$$

We shall write (u, v, w) for $\left(\frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt}\right)$. Then from (10), (11), and (12), the components $(\lambda^0, \lambda^1, \lambda^2, \lambda^3)$ can be written

$$\left[\left(\frac{X}{t_0^2}\right)^n\frac{1}{Y^{\frac{1}{2}}}, \quad \left(\frac{X}{t_0^2}\right)^n\frac{u}{Y^{\frac{1}{2}}}, \quad \left(\frac{X}{t_0^2}\right)^n\frac{v}{Y^{\frac{1}{2}}}, \quad \left(\frac{X}{t_0^2}\right)^n\frac{w}{Y^{\frac{1}{2}}}\right].$$

* *Proc. Edinburgh Math. Soc.* **4** (1936), 238.

† Walker (1936).

Hence

$$\lambda_0 = \bar{g}_{00}\lambda^0 = \left(\frac{t_0^2}{X}\right)^n \frac{1}{Y^{\frac{1}{2}}}, \qquad \frac{\partial(\lambda^1,\lambda^2,\lambda^3)}{\partial(u,v,w)} = \left(\frac{X}{t_0^2}\right)^{3n} \frac{dudvdw}{Y^{\frac{3}{2}}}.$$

Therefore
$$d\Omega = \frac{dudvdw}{\{1-(u^2+v^2+w^2)/c^2\}^2}. \tag{15}$$

This is an invariant expression for all observers, and a constant multiple of $d\Omega$ (actually B/c^3) gives the velocity distribution of fundamental particles occurring in Milne's theory.*

The element of volume of a three-dimensional space orthogonal to λ^i at P, is given by†

$$dV = (-\bar{g})^{\frac{1}{2}}\lambda^0\, dx^1dx^2dx^3 = \left(\frac{t_0^2}{X}\right)^{3n} \frac{dxdydz}{Y^{\frac{1}{2}}}. \tag{16}$$

Hence a system of fundamental particles having the number density ν and velocity distribution $\mathbf{V} = \mathbf{P}/t$ will have a space distribution

$$\nu\, dV = \nu \left(\frac{t_0^2}{X}\right)^{3n} \frac{t\, dxdydz}{X^{\frac{1}{2}}}. \tag{17}$$

This gives Milne's distribution of fundamental particles

$$\frac{Bt\, dxdydz}{c^3\{t^2-(x^2+y^2+z^2)/c^2\}^2} \tag{18}$$

provided that
$$\nu = \frac{B}{c^3t_0^3}\left(\frac{X}{t_0^2}\right)^{3(n-\frac{1}{2})}. \tag{19}$$

In the particular case $n = \frac{1}{2}$, the particle density has the value $B/c^3t_0^3$. In the space-time of the substratum, therefore, the density-distribution of fundamental particles is stationary and homogeneous.‡ This result has been obtained by Milne.§

5. A statistical system of particles is defined by a velocity-distribution function $\phi(x,\lambda)$ such that the number of particles crossing the volume element dV whose paths issue in the angle $d\Omega$ is given by‖

$$dN = \phi(x,\lambda)\, dV d\Omega. \tag{20}$$

In order that particle number shall be conserved, $\phi(x,\lambda)$ must satisfy the generalized Boltzmann equation

$$\frac{\partial\phi}{\partial x^i}\lambda^i - \frac{\partial\phi}{\partial\lambda^i}\bar{\Gamma}^i_{jk}\lambda^j\lambda^k = 0, \tag{21}$$

* *W.S.* 87. † Walker (1936). ‡ *W.S.* 64.
§ *Proc. Royal Soc.* A, **159** (1937), 171. ‖ Walker (1936).

where the Christoffel symbols are given by*

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk}+\delta^i_j\frac{\partial\sigma}{\partial x_k}+\delta^i_k\frac{\partial\sigma}{\partial x_j}-g_{jk}\,g^{il}\frac{\partial\sigma}{\partial x^l} \tag{22}$$

and the symbols Γ^i_{jk} for a flat space are zero.

We define dimensionless quantities χ, θ by

$$\chi = \frac{X}{t_0^2} = \frac{g_{hk}\,x^h x^k}{t_0^2}, \qquad \theta = \frac{1}{2t_0}\frac{dX}{ds} = \frac{g_{hk}\,x^h\lambda^k}{t_0} \tag{23}$$

and we look for solutions of (21) of the form

$$\phi = c^{-6}t_0^{-3}\Phi(\chi,\theta). \tag{24}$$

This function Φ must be dimensionless in order to make the right-hand side of (20) a pure number.

From (10), (22), (23) we find

$$\bar{\Gamma}^i_{jk}\lambda^j\lambda^k = -\frac{4n\theta}{\chi t_0}\lambda^i+\frac{2n\chi^{2n-1}}{t_0^2}x^i, \tag{25}$$

$$\frac{\partial\Phi}{\partial x^i}\lambda^i = \frac{1}{t_0}\left(2\theta\frac{\partial\Phi}{\partial\chi}+\chi^{2n}\frac{\partial\Phi}{\partial\theta}\right),$$

$$\frac{\partial\Phi}{\partial\lambda^i}\bar{\Gamma}^i_{jk}\lambda^j\lambda^k = \frac{1}{t_0}\left(-\frac{4n\theta^2}{\chi}+2n\chi^{2n}\right)\frac{\partial\Phi}{\partial\theta}.$$

Hence equation (21) reduces to

$$2\theta\frac{\partial\Phi}{\partial\chi}+\left\{\frac{4n\theta^2}{\chi}-(2n-1)\chi^{2n}\right\}\frac{\partial\Phi}{\partial\theta} = 0. \tag{26}$$

This has solutions of the form

$$\Phi = f\left(\frac{\theta^2}{\chi^{4n}}-\frac{1}{\chi^{2n-1}}\right) \equiv f(\zeta) \quad \text{say}, \tag{27}$$

where, with the notation of § 2,

$$\frac{\theta^2}{\chi^{2n+1}} = \xi, \qquad \zeta = (\xi-1)\left(\frac{t_0^2}{X}\right)^{2n-1}. \tag{28}$$

Then, from (15), (16), (24), the number distribution (20) becomes

$$dN = f(\zeta)\frac{dxdydzdudvdw}{c^6t_0^3(X/t_0^2)^{3n}Y^{\frac{3}{2}}}. \tag{29}$$

We can now find the statistical distributions which are independent of the arbitrary unit t_0. Since $f(\zeta)$ is dimensionless, (29) will, in

* Cf. L. P. Eisenhart, *Riemannian Geometry* (1926), 89.

general, only be independent of t_0 when $f(\zeta) = C_n \zeta^{-\frac{3}{2}}$, where C_n is a dimensionless constant. We then have

$$dN = \frac{C_n}{c^6(\xi-1)^{\frac{3}{2}}} \frac{dxdydzdudvdw}{X^{\frac{3}{2}}Y^{\frac{3}{2}}} \tag{30}$$

Comparing this with (5) we see that, in general, the statistical systems of general relativity, in space-time having a metric given by (1) and (8), correspond to the systems of Milne's theory having

$$\psi(\xi) = \frac{\text{constant}}{(\xi-1)^{\frac{3}{2}}}. \tag{31}$$

In the particular case $n = \frac{1}{2}$, however, the number distribution (29) can take the general form (5) of the distribution in Milne's theory. This gives the distribution of free particles in the space-time of the substratum.

We therefore conclude that, in a space-time in which a set of fundamental particles having a hydrodynamic motion can be constructed, it is also possible to construct statistical systems of particles defined by the distribution functions (31), except in the particular case when the metric of space-time has the form (9). In the latter case the form of $\psi(\xi)$ is not uniquely determined. We shall now show that constrained systems of particles, having the properties of Milne's statistical systems, can be constructed in the space-time of the substratum. The argument is essentially similar to that given by Milne.*

6. The particles of the statistical systems just considered were following geodesic paths, and in the space-time of the substratum it can be shown that ξ = constant along a geodesic path. We now suppose that each particle of a statistical system, in this particular space-time, is constrained to follow some path along which ξ varies. We therefore introduce into the geodesic equations an acceleration four-vector F^i, such that

$$\frac{d\lambda^i}{d\tau} + \bar{\Gamma}^i_{jk}\lambda^j\lambda^k = F^i, \tag{32}$$

where $\bar{\Gamma}^i_{jk}\lambda^j\lambda^k$ is given by (25) when $n = \frac{1}{2}$ and $\lambda^i = dx^i/d\tau$. From (9), (23), (25) this equation reduces to

$$\frac{X^{\frac{1}{2}}}{Y^{\frac{1}{2}}}\frac{d}{dt}\left\{\frac{X^{\frac{1}{2}}}{Y^{\frac{1}{2}}}\frac{dx^i}{dt}\right\} + \left(x^i - \frac{2Z}{Y}\frac{dx^i}{dt}\right)\frac{1}{t_0^2} = F^i.$$

* *W.S.* 179.

This can also be written

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{1}{Y^{\frac{1}{2}}}\frac{dx^i}{dt}\right)+\left(x^i-\frac{Z}{Y}\frac{dx^i}{dt}\right)\frac{1}{X}=\frac{t_0^2F^i}{X}. \tag{33}$$

The form of F^i must now be chosen so that (33) satisfies the cosmological principle, that is to say, is invariant under a transformation from the coordinate system of one observer to that of any other observer. According to Milne, equation (2) gives the general form of the equation of motion of a particle satisfying this principle. This equation can be written in the form

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{\mathbf{V}}{Y^{\frac{1}{2}}}\right)+\left(\mathbf{P}-\frac{\mathbf{V}Z}{Y}\right)\frac{1}{X}=[1+G(\xi)]\left(\mathbf{P}-\frac{\mathbf{V}Z}{Y}\right)\frac{1}{X}. \tag{34}$$

We also associate with (34) the time-component equation

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{c}{Y^{\frac{1}{2}}}\right)+\left(ct-\frac{cZ}{Y}\right)\frac{1}{X}=[1+G(\xi)]\left(ct-\frac{cZ}{Y}\right)\frac{1}{X}. \tag{35}$$

Comparing these equations with (33), we see that

$$F^i=\frac{[1+G(\xi)]}{t_0^2}\left(x^i-\frac{Z}{Y}\frac{dx^i}{dt}\right).$$

Using (9) and (11) this can be written

$$F^i=\frac{[1+G(\xi)]}{t_0^2}\left(\frac{g_{lm}\lambda^l\lambda^m x^i}{X}-\frac{g_{lm}\lambda^i\lambda^l x^m}{X}\right)$$

$$=[1+G(\xi)]\frac{g_{lm}\lambda^l}{X}(\lambda^m x^i-\lambda^i x^m). \tag{36}$$

7. We can now construct the constrained systems of particles having the space-velocity distributions (5), in the space-time of the substratum. The distribution function $\psi(\xi)$ must satisfy a certain generalized Boltzmann equation in order that particle number shall be conserved. When the equation of motion of the particles is given by (32), this equation has been found to be*

$$\frac{\partial\psi(\xi)}{\partial x^i}\lambda^i-\frac{\partial\psi(\xi)}{\partial\lambda^i}\bar{\Gamma}^i_{jk}\lambda^j\lambda^k+\frac{\partial}{\partial\lambda^i}\{\psi(\xi)F^i\}=0. \tag{37}$$

From (32) this can be written

$$\frac{d\psi(\xi)}{d\tau}+\psi(\xi)\frac{\partial F^i}{\partial\lambda^i}=0. \tag{38}$$

* This equation was obtained for me by Dr. T. G. Cowling. It can be derived by the method used in Walker's paper (1936), on the Boltzmann equation, if account be taken of the additional term F^i in the equations of motion.

For the space-time of the substratum

$$\xi = \frac{(g_{hk}x^h\lambda^k)^2t_0^2}{X^2}, \qquad \frac{\partial\xi}{\partial\lambda^i} = \frac{2g_{ij}x^j(g_{hk}x^h\lambda^k)t_0^2}{X^2},$$

so that from (36)

$$\frac{\partial F^i}{\partial\lambda^i} = -\frac{2g_{lm}\lambda^l x^m}{X}\left[\frac{dG(\xi)}{d\xi}(\xi-1)+\tfrac{3}{2}\{1+G(\xi)\}\right]. \tag{39}$$

Also, since (33) reduces to (2), it can be shown that

$$\frac{d\xi}{dt} = -\frac{2Z}{X}(\xi-1)[1+G(\xi)],$$

whence $$\frac{d\psi(\xi)}{d\tau} = -\frac{2g_{lm}\lambda^l x^m}{X}\frac{d\psi(\xi)}{d\xi}(\xi-1)[1+G(\xi)]. \tag{40}$$

From (39) and (40) equation (38) becomes

$$-\frac{2g_{lm}\lambda^l x^m}{X}\times$$

$$\times\left[\frac{d\psi(\xi)}{d\xi}(\xi-1)\{1+G(\xi)\}+\psi(\xi)(\xi-1)\frac{dG(\xi)}{d\xi}+\tfrac{3}{2}\psi(\xi)\{1+G(\xi)\}\right] = 0.$$

Therefore $$\frac{1}{\psi(\xi)}\frac{d\psi(\xi)}{d\xi}+\frac{1}{1+G(\xi)}\frac{dG(\xi)}{d\xi}+\frac{3}{2(\xi-1)} = 0. \tag{41}$$

This equation integrates to give Milne's relation (6).

8. We conclude that the substratum can be completely described by the space-time (9), in accordance with general relativity. It is also worth pointing out that our work has shown that a system of free particles having the general distribution function $\psi(\xi)$ can only be constructed in a space-time having the metric (9). This is in agreement with Milne's conclusion that $G(\xi) \equiv -1$ defines a pure substratum. We also observe that Walker's result concerning particle-distribution in general relativity spaces agrees, as a special case, with Milne's general relation between $G(\xi)$ and $\psi(\xi)$.

The metric (9), besides defining the space-time of a pure substratum, gives the interval $d\tau$ for a universe. In the latter case, however, the tracks of particles are not given by the variational principle $\delta\int d\tau = 0$, but depend on a constant C which is a measure of the mass of the particles.

The interval of proper time of an observer in the universe is, from (9),

$$d\tau = \pm t_0\,dt/t. \tag{42}$$

Here t is the 'kinematic time' of the observer which we shall assume to be increasing with τ, and to have the value t_0 when $\tau = t_0$. Then, integrating (42), we find

$$\tau - t_0 = t_0 \log t/t_0. \tag{43}$$

This is the relation Milne has found* between the 'kinematic time' and the 'Newtonian time' of an observer, when t_0 is the time at which the clocks graduated on these different scales agree. The latter has been called the 'present time'. The 'Newtonian time' of an observer is therefore the same as his 'proper time', and the constant t_0 is a measure of the present age of the universe on the kinematic scale.

9. Summary. A description is given of Milne's systems of particles according to general relativity. The metric of space-time is one for which the geodesic paths give the trajectories of free particles in the substratum. The acceleration function for the particles of a statistical system has the same form as that given by the kinematic treatment. The 'proper time' of an observer agrees with his 'Newtonian time'.

* *Proc. Royal Soc.* A, **158** (1937), 324.

ON DIRICHLET SERIES WHICH SATISFY A CERTAIN FUNCTIONAL EQUATION

By H. HEILBRONN (Cambridge)

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WE consider a Dirichlet series

$$\phi(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{1}$$

which satisfies the following four conditions:

(i) the series (1) has a half-plane of convergence, so that

$$f_\mu(\gamma) = \sum_{n=1}^{\infty} a_n e^{-n^\mu \gamma}$$

is absolutely convergent for $\gamma > 0$, $\mu > 0$;

(ii) $\phi(s)$ is regular for all finite values of s apart from a finite number of poles;

(iii) we can find a positive integer k and positive numbers A and λ such that

$$\Phi(s) = A^{-s}\Gamma^k(s/\lambda)\phi(s)$$

satisfies

$$\Phi(s) = \Phi(1-s); \tag{2}$$

(iv)

$$\phi(s) = O(|t|^P) \tag{3}$$

uniformly for $\sigma \geqslant \frac{1}{2}$, $|t| \to \infty$, where the constant P depends on the function only.*

By $\Phi_0(s)$ we denote the rational function which vanishes at infinity and makes $\Phi(s)-\Phi_0(s)$ an integral function; $x_1,\ldots, x_k$ are positive variables; we denote their sum by $S(x)$ and their product by $N(x)$; μ is an abbreviation for λ/k.

Then we have for all values of s

$$\Phi(s) = \Phi_0(s) + \int\limits_{N(x)>1}\cdots\int \{N(x)^{s/\lambda}+N(x)^{(1-s)/\lambda}\} f_\mu\{A^\mu S(x)\}\frac{dx_1}{x_1}\cdots\frac{dx_k}{x_k}. \tag{4}$$

The proof of (4) is easy. It follows from (3) that the difference of the two sides of (4) is regular for all s and bounded if σ is bounded. Hence it suffices to show that the difference tends to zero as σ tends to $+\infty$ and $-\infty$. But (2) shows that all the three terms in (4) are

* This condition can be replaced by a weaker one by the use of an argument of Phragmén-Lindelöf.

even functions of $s-\frac{1}{2}$. Hence we need investigate the case $\sigma \to +\infty$ only.

For sufficiently large positive σ the series in (1) is absolutely convergent. Hence we have for $n > 0$

$$A^{-s/k}\Gamma(s/\lambda)n^{-s/k} = \int_0^\infty x^{s/\lambda}e^{-A^\mu n^\mu x}\frac{dx}{x},$$

$$A^{-s}\Gamma^k(s/\lambda)n^{-s} = \int\limits_{N(x)>0}\cdots\int N(x)^{s/\lambda}e^{-A^\mu n^\mu S(x)}\frac{dx_1}{x_1}\cdots\frac{dx_k}{x_k},$$

$$\Phi(s) = \int\limits_{N(x)>0}\cdots\int N(x)^{s/\lambda}f_\mu\{A^\mu S(x)\}\frac{dx_1}{x_1}\cdots\frac{dx_k}{x_k}.$$

But the two integrals

$$\int\limits_{0\leqslant N(x)\leqslant 1}\cdots\int N(x)^{s/\lambda}f_\mu\{A^\mu S(x)\}\frac{dx_1}{x_1}\cdots\frac{dx_k}{x_k}$$

and

$$\int\limits_{N(x)>1}\cdots\int N(x)^{(1-s)/\lambda}f_\mu\{A^\mu S(x)\}\frac{dx_1}{x_1}\cdots\frac{dx_k}{x_k}$$

tend to zero as $\sigma \to \infty$. So does $\Phi_0(s)$, and (4) is proved.

We mention one application of (4).

Let d_1, d_2, d_3 be discriminants of quadratic fields and let $L_{d_i}(s)$ denote the series

$$\sum_{n=1}^\infty (d_i/n)n^{-s}.$$

Suppose further that $d_1 d_2 d_3$ is a perfect square. Then

$$\phi(s) = \zeta(s)\prod_{i=1}^3 L_{d_i}(s)$$

has non-negative coefficients and satisfies our conditions (i)–(iv) with

$$k = \tfrac{1}{2}\Big(5+\sum_{i=1}^3 \operatorname{sign} d_i\Big),$$
$$\lambda = \tfrac{1}{2}k,$$
$$A = 2^{4-k}\pi^2\prod_{i=1}^3 |d_i|^{-\frac{1}{2}};$$

and (4) is the identity from which Siegel* derived his important result

$$\log L_d(1) = o(\log|d|).$$

* *Acta Arithmetica*, 1, 83–6.

ON CERTAIN HANKEL TRANSFORMS

By A. ERDÉLYI (*Brno*)

[Received 28 January 1938]

1. In a recent paper* R. S. Varma has developed the Hankel transform of the function $e^{-kx}x^{\frac{1}{2}\nu}L_n^{(\nu)}(x)$ in a series of Laguerre polynomials

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{m!}$$

and proved Wilson's integral equation† to be a special case of his formulae. It is the purpose of the present note to point out that term-by-term computation of the Hankel transform of any function of the form $e^{-kx}x^{\mu}P(x)$, $P(x)$ being *any* polynomial, yields a finite series of Whittaker's confluent hypergeometric functions. If $2\mu = \nu$ (the order of the Hankel transform), this expansion reduces to a series of Laguerre polynomials. In the special case

$$P(x) = L_n^{(\alpha)}(x) \quad \text{and} \quad 2\mu = \nu = \alpha$$

the series obtained by this can be summed by means of the multiplication formula of Laguerre polynomials. The integral equations satisfied by Laguerre polynomials and products of such polynomials are special cases of this formula.

2. We denote by
$$P(z) = \sum_{m=0}^{n} c_m z^m \tag{1}$$

any polynomial, and compute the Hankel transform of order ν of the function
$$F(x) = e^{-kx}x^{\mu}P(x), \tag{2}$$

i.e. the infinite integral
$$T = \int_0^\infty J_\nu\{2\surd(xy)\}F(y)\,dy, \tag{3}$$

putting (2) in (3) and integrating term by term. Thus we obtain
$$T = \sum_{m=0}^{n} c_m \int_0^\infty J_\nu\{2\surd(xy)\}e^{-ky}y^{\mu+m}\,dy. \tag{4}$$

* R. S. Varma, *Acta pontif. Acad. Sci.* 1 (1937), 37–41.

† B. M. Wilson, *Messenger of Math.* 53 (1923/4), 157–60.

Using Hankel's generalization of 'Weber's first exponential integral'* in the form

$$\int_0^\infty e^{-ky}y^\lambda J_\nu\{2\sqrt{(xy)}\}\,dy = \frac{\Gamma(\lambda+\frac{1}{2}\nu+1)}{\Gamma(\nu+1)}k^{-\lambda-\frac{1}{2}}x^{-\frac{1}{2}}e^{-\frac{1}{2}(x/k)}M_{\lambda+\frac{1}{2},\frac{1}{2}\nu}\left(\frac{x}{k}\right)$$

$$[\Re(k) > 0,\ \Re(\lambda+\tfrac{1}{2}\nu) > -1],$$

we obtain at once the expansion

$$\int_0^\infty e^{-ky}y^\mu P(y)J_\nu\{2\sqrt{(xy)}\}\,dy \tag{5}$$

$$= \frac{1}{\Gamma(\nu+1)}k^{-\mu-\frac{1}{2}}x^{-\frac{1}{2}}e^{-\frac{1}{2}(x/k)}\sum_{m=0}^{n} c_m\,\Gamma(\mu+\tfrac{1}{2}\nu+m+1)k^{-m}M_{\mu+m+\frac{1}{2},\frac{1}{2}\nu}\left(\frac{x}{k}\right)$$

$$[\Re(k) > 0,\ \Re(\mu+\tfrac{1}{2}\nu) > -1].$$

This expansion is also valid for power series $P(x)$ convergent for all finite values of $|x|$, supposing that the integral on the left and the infinite series on the right converge absolutely.

3. In (5) we put $\mu = \frac{1}{2}\nu$ and observe that in this case Whittaker's functions can be expressed in terms of Laguerre polynomials, for

$$M_{\frac{1}{2}\nu+m+\frac{1}{2},\frac{1}{2}\nu}(z) = \frac{m!\,\Gamma(\nu+1)}{\Gamma(m+\nu+1)}z^{\frac{1}{2}\nu+\frac{1}{2}}e^{-\frac{1}{2}z}L_m^{(\nu)}(z).$$

Thus we obtain

$$\int_0^\infty e^{-ky}y^{\frac{1}{2}\nu}P(y)J_\nu\{2\sqrt{(xy)}\}\,dy = k^{-\nu-1}x^{\frac{1}{2}\nu}e^{-x/k}\sum_{m=0}^{n} m!\,c_m\,k^{-m}L_m^{(\nu)}\left(\frac{x}{k}\right).$$

$$[\Re(k) > 0,\ \Re(\nu) > -1]. \tag{6}$$

The expansion of the Hankel transform in a series of Whittaker functions and Laguerre polynomials respectively is intelligible by the fact that these functions are characteristic functions of the Hankel kernel, the corresponding characteristic numbers (the only ones of the Hankel kernel) being ± 1, each of infinite order.

4. Some special cases, in which the series on the right of (5) and (6) can be summed by means of known relations, may be considered. First put

$$P(x) = L_n^{(\mu+\frac{1}{2}\nu)}(kx), \qquad c_m = \frac{(-n)_m\,\Gamma(\mu+\frac{1}{2}\nu+n+1)}{m!\,n!\,\Gamma(\mu+\frac{1}{2}\nu+m+1)}k^m,$$

* G. N. Watson, *Bessel Functions* (1922), § 13.3 (3). See also A. Erdélyi, *Math. Annalen*, 113 (1936), 357–62.

and make use of the relation*

$$\sum_{m=0}^{n} \frac{(-n)_m}{m!} M_{\mu+m+\frac{1}{2},\frac{1}{2}\nu}(z) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+n+1)} z^{\frac{1}{2}n} M_{\mu+\frac{1}{2}n+\frac{1}{2},\frac{1}{2}\nu+\frac{1}{2}n}(z),$$

thus obtaining

$$\int_0^\infty e^{-ky} y^{\mu} J_\nu\{2\sqrt{(xy)}\} L_n^{(\mu+\frac{1}{2}\nu)}(ky)\, dy \tag{7}$$

$$= \frac{\Gamma(\mu+\frac{1}{2}\nu+n+1)}{\Gamma(\nu+n+1)} k^{-\mu-\frac{1}{2}n-\frac{1}{2}} x^{\frac{1}{2}n-\frac{1}{2}} e^{-\frac{1}{2}(x/k)} M_{\mu+\frac{1}{2}n+\frac{1}{2},\frac{1}{2}\nu+\frac{1}{2}n}\left(\frac{x}{k}\right)$$

$$[\Re(k) > 0,\ \Re(\mu+\tfrac{1}{2}\nu) > -1],$$

an integral representation of Whittaker's function, which is equivalent to a generating function of the confluent hypergeometric function.† Next replace x and y by kx and y/k respectively and put in (6)

$$P(x) = L_n^{(\alpha)}(x), \qquad c_m = \binom{n+\alpha}{n-m}\frac{(-)^m}{m!}, \qquad \nu = \alpha.$$

Using the multiplication formula of Laguerre polynomials‡

$$\sum_{m=0}^{n} \binom{n+\alpha}{n-m} (-k)^{n-m} L_m^{(\alpha)}(z) = (1-k)^n L_n^{(\alpha)}\left(\frac{z}{1-k}\right),$$

we get at once the relation

$$\int_0^\infty e^{-y} y^{\frac{1}{2}\alpha} J_\alpha\{2\sqrt{(xy)}\} L_n^{(\alpha)}\left(\frac{y}{k}\right) dy = \left(1-\frac{1}{k}\right)^n x^{\frac{1}{2}\alpha} e^{-x} L_n^{(\alpha)}\left(\frac{x}{1-k}\right) \tag{8}$$

$$[\Re(\alpha) > -1].$$

Wilson's integral equation is the special case $k = \frac{1}{2}$ of this formula. The limiting cases $k \to 0$ and $k \to 1$ of (8) yielding Le Roy's integral representation of Laguerre polynomials§ and its inversion, the formula for the Hankel transform of the function $e^{-x} x^{\frac{1}{2}\nu} L_n^{(\nu)}(x)$ respectively, are also known.

* A. Erdélyi, *Math. Zeits.* 42 (1937), 641–70 equation (1, 1).

† A. Erdélyi, *Monatsh. f. Math. u. Phys.* 46 (1937), 132–56 equation (7, 3).

‡ A. Erdélyi, *Math. Zeits.* 42 (1936), 125–43 equation (5, 4).

§ E. Le Roy, *Annales de Toulouse* (2) 2 (1900), 317–430. Cf. especially 379–84. See also G. Szegö, *Math. Zeits.* 25 (1926), 87–115.

ON WARING'S PROBLEM

By LOO-KENG HUA (*Cambridge*)

[Received 1 February 1938]

THE object of the present paper is to give a proof that Hardy and Littlewood's asymptotic formula for the number of solutions of the Diophantine equation

$$N = x_1^k + \ldots + x_s^k \quad (x_\nu \geqslant 0)$$

is true for $s \geqslant 2^k+1$. This result is new only for $k < 14$. For $k \geqslant 14$, Vinogradow's contribution is much better than this. The most interesting particular case is $k = 4$, for which Estermann† and Davenport and Heilbronn‡ proved that every sufficiently large integer is a sum of 17 fourth powers, but they gave no asymptotic formula for the number of solutions.

More precisely, what I am going to prove is the following more general theorem:

Let $P_1(x),\ldots, P_s(x)$ be s integral-valued polynomials of the kth degree and let their first coefficients be positive numbers $a_1,\ldots, a_s$ respectively. Let $r(N)$ be the number of solutions of the Diophantine equation

$$N = P_1(x_1) + \ldots + P_s(x_s) \quad (x_\nu \geqslant 0).$$

Then, for $s \geqslant 2^k+1$, we have

$$r(N) = \prod_{\nu=1}^{s} a_\nu^{-1/k} \frac{\Gamma^s(1+1/k)}{\Gamma(s/k)} \mathfrak{S}(N) N^{s/k-1} + O(N^{s/k-1-\delta}),$$

where $\delta = 2^{1-k}s - z - \epsilon$ and ϵ is an arbitrary small positive number and $\mathfrak{S}(N)$ is defined in one of my previous papers.§

I shall give elsewhere an application of this theorem to prove that

$$G\{P(x)\} \leqslant 17,$$

provided that $P(x)$ is a quartic polynomial with coefficient of x^4 positive and that there does not exist an integer q (> 1) such that

$$P(x) \equiv P(0) \pmod q$$

for all x.

The proof of the theorem depends essentially on the following lemma which seems to have some interest in itself.

† *Proc. London Math. Soc.* 41 (1938), 127–42.

‡ Ibid. 41 (1938), 143–50.

§ Ibid. 43 (1937), 161–82.

MAIN LEMMA. *Let $P(x)$ be an integral-valued polynomial of the kth degree, and*

$$f(\alpha) = \sum_{x=1}^{p} \exp\{2\pi i\, P(x)\alpha\}.$$

Then

$$\int_0^1 |f(\alpha)|^\lambda \, d\alpha = O(p^{\mu(\lambda)}),$$

where $\{\lambda, \mu(\lambda)\}$ lies on a polygonal line with vertices $(2^\nu, 2^\nu - \nu + \epsilon)$ $(\nu = 1,..., k)$, and the constants implied by the symbol O depend only on the coefficients of $P(x)$ and ϵ.

An improvement of this lemma for the particular case $P(x) = x^k$ and its application to the additive prime-number theory will appear elsewhere later.

Proof of the main lemma

Since

$$\log\left(\int_0^1 |f(\alpha)|^\nu \, d\alpha\right)$$

is a convex function of ν,† we only need to prove that

$$\int_0^1 |f(\alpha)|^{2^\nu} \, d\alpha = O(p^{2^\nu - \nu + \epsilon}) \quad \text{for} \quad \nu = 1, 2,..., k. \tag{1}$$

Without loss of generality we assume that $P(x)$ is a polynomial with integer coefficients. In fact, let q be the least common denominator of the coefficients of $P(x)$, then, by Hölder's inequality,

$$\int_0^1 |f(\alpha)|^\lambda \, d\alpha = \int_0^1 \left| \sum_{a=1}^{q} \sum_{x=0}^{[(p-a)/q]} \exp\{2\pi i\, P(qx+a)\alpha\} \right|^\lambda d\alpha$$

$$\leqslant q^{\lambda-1} \sum_{a=1}^{q} \int_0^1 \left| \sum_{x=0}^{[(p-a)/q]} \exp[2\pi i\{P(qx+a) - P(a)\}\alpha] \right|^\lambda d\alpha,$$

where $P(qx+a) - P(a)$ is a polynomial with integer coefficients.

Then (1) is trivial for $\nu = 1$ and well known for $\nu = 2$.§ We are going to prove (1) by induction.

We use the abbreviation

$$\underset{y}{\Delta}\, Q(x) = \frac{1}{y}\{Q(x+y) - Q(x)\}.$$

† See, for example, Hardy, Littlewood, and Pólya, *Inequalities*, § 6.12.

§ See, for example, Landau, *Vorlesungen über Zahlentheorie*, Bd. 1, Satz 262, 37. There he deals only with the particular case $P(x) = x^k$. For the general case see Hua, *J. of Chinese Math. Soc.* 1 (1936), 23–61, Lemma 11.

Then $\underset{y}{\Delta} Q(x)$ is a polynomial of the $(h-1)$th degree in x, provided that $Q(x)$ is a polynomial of the hth degree. Let $\sum_{x}^{p}$ denote a summation with variable x whose number of terms is $O(p)$.

Consider

$$|f(\alpha)|^2 = \sum_{x_1=1}^{p} \sum_{x_2=1}^{p} \exp[2\pi i\{P(x_1)-P(x_2)\}\alpha]$$
$$= \sum_{x_2}^{p} \sum_{y_1}^{p} \exp[2\pi i\{P(x_2+y_1)-P(x_2)\}\alpha]$$
$$= \sum_{y_1}^{p} \sum_{x_2}^{p} \exp\{2\pi i y_1 \underset{y_1}{\Delta} P(x_2)\alpha\}.$$

By Schwarz's inequality, we have

$$|f(\alpha)|^4 \ll p \sum_{y_1}^{p} \Big| \sum_{x_2}^{p} \exp\{2\pi i y_1 \underset{y_1}{\Delta} P(x_2)\alpha\}\Big|^2$$
$$\ll p \sum_{y_1}^{p} \sum_{y_2}^{p} \sum_{x_3}^{p} \exp\{2\pi i y_1 y_2 \underset{y_2}{\Delta} \underset{y_1}{\Delta} P(x_3)\alpha\},$$

where $A \ll B$ means $A = O(B)$. Repeating this process, we obtain, in general,

$$|f(\alpha)|^{2^\mu} \ll p^{2^\mu-\mu-1} \sum_{y_1}^{p} \dots \sum_{y_\mu}^{p} \sum_{x_{\mu+1}}^{p} \exp\{2\pi i y_1 \dots y_\mu \underset{y_\mu}{\Delta} \underset{y_{\mu-1}}{\Delta} \dots \underset{y_1}{\Delta} P(x_{\mu+1})\alpha\}$$
$$\ll p^{2^\mu-1}+p^{2^\mu-\mu-1} \sum_{y_1}^{p} \dots \sum_{y_\mu}^{p} \sum_{x_{\mu+1}}^{p} {}^{*} \exp\{2\pi i y_1 \dots y_\mu \underset{y_\mu}{\Delta} \dots \underset{y_1}{\Delta} P(x_{\mu+1})\alpha\} \quad (2)$$

for $\mu = 1, 2,\dots, k-1$, where * denotes the condition

$$y_1 \cdots y_\mu \underset{y_\mu}{\Delta} \dots \underset{y_1}{\Delta} P(x_{\mu+1}) \neq 0.$$

We have, therefore,

$$\int_0^1 |f(\alpha)|^{2^\nu}\, d\alpha \ll p^{2^{\nu-1}-1} \int_0^1 |f(\alpha)|^{2^{\nu-1}}\, d\alpha + \quad (3)$$
$$+p^{2^{\nu-1}-\nu} \int_0^1 \sum_{y_1}^{p} \dots \sum_{y_{\nu-1}}^{p} \sum_{x_\nu}^{p} {}^{*} \exp\{2\pi i y_1 \dots y_{\nu-1} \underset{y_{\nu-1}}{\Delta} \dots \underset{y_1}{\Delta} P(x_\nu)\alpha\}|f(\alpha)|^{2^{\nu-1}}\, d\alpha.$$

By the hypothesis of the induction, the first term on the right of (3) is

$$O(p^{2^{\nu-1}-1}.p^{2^{\nu-1}-\nu+1+\epsilon}) = O(p^{2^\nu-\nu+\epsilon}).$$

The second term on the right of (3) equals

$$p^{2^{\nu-1}-\nu} \int_0^1 \sum_{y_1}^{p} \dots \sum_{y_\mu}^{p} \sum_{x_{\mu+1}}^{p} {}^{*} \sum_{z_1}^{p} \dots \sum_{z_{2^{\nu-1}}}^{p} \exp[2\pi i\{y_1 \dots y_{\nu-1} \underset{y_{\nu-1}}{\Delta} \dots \underset{y_1}{\Delta} P(x_\nu)-$$
$$-P(z_1)+P(z_2)-\dots+P(z_{2^{\nu-1}})\}\alpha]\, d\alpha = p^{2^{\nu-1}-\nu}R,$$

where R denotes the number of solutions of

$$y_1 \cdots y_{\nu-1} \underset{y_{\nu-1}}{\Delta} \underset{y_1}{\Delta} \cdots P(x_\nu) = P(z_1) - P(z_2) + \ldots - P(z_{2^{\nu-1}}),$$

$$y_1 \cdots y_{\nu-1} \underset{y_{\nu-1}}{\Delta} \cdots \underset{y_1}{\Delta} P(x_\nu) \neq 0, \qquad z_\mu,\, y_\mu,\, x_{\nu+1} \ll P. \tag{4}$$

For given $z_1, \ldots, z_{2^{\nu-1}}$, the number of solutions of (4) is

$$O\{d^{\nu-1}[P(z_1) - P(z_2) + \ldots - P(z_{2^{\nu-1}})]\}.\dagger$$

Since $d(n) = O(n^\epsilon)$, we have

$$R \ll \sum_{z_1} \cdots \sum_{z_{2^{\nu-1}}} \ddagger\, d^{\nu-1}\{P(z_1) - P(z_2) + \ldots - P(z_{2^{\nu-1}})\}$$

$$\ll p^{2^{\nu-1}+\epsilon},$$

where $\ddagger$ denotes the condition $P(z_1) - P(z_2) + \ldots - P(z_{2^{\nu-1}}) \neq 0$. The lemma is therefore proved.

Proof of the theorem

Let $p = N^{1/k}$ and

$$S_r(\alpha) = \sum_{x=1}^{p_r} \exp\{2\pi i\, P_r(x)\alpha\},$$

where p_r is the greatest root of $P_r(x) = N$. It is evident that when N is sufficiently large, p_r always exists and $p \ll p_r \ll p$. Then

$$r(N) = \int_0^1 \prod_{r=1}^{s} S_r(\alpha) \exp(-2\pi i \alpha N)\, d\alpha.$$

It is sufficient to estimate the part $\overline{W}$ of the integral corresponding to the minor arcs, since the remaining part can be treated, without any difficulty, by the method used in my previous paper.§

By Weyl's theorem‖ and the main lemma we have

$$\overline{W} \ll p^{(1-2^{1-k}+\epsilon)(s-2^k)} \int_0^1 \prod_{r=1}^{2^k} |S_r(\alpha)|\, d\alpha$$

$$\ll p^{(1-2^{1-k}+\epsilon)(s-2^k)} \Big(\prod_{r=1}^{2^k} \int_0^1 |S_r(\alpha)|^{2^k}\, d\alpha \Big)^{2^{-k}}$$

$$\ll p^{s-k-\delta}.$$

The theorem is proved.

In closing, I should like to express my warmest thanks to the referee for his valuable advice.

† $d(n)$ denotes number of divisors of n.

§ *Proc. London Math. Soc.* 43 (1937), 161–82.

‖ Landau, *Vorlesungen über Zahlentheorie*, Satz 267.

A PAIR OF FUNCTIONS WHICH ARE FOURIER SINE-TRANSFORMS OF EACH OTHER

By R. S. VARMA (*Cawnpore*)

[Received 22 February 1938]

THE object of this paper is to establish the following

THEOREM. *If*

$$\phi(x) = i\{D^2_{-(n+1)}(ix) - D^2_{-(n+1)}(-ix)\}$$

and

$$\psi(x) = (-)^n \frac{2\pi}{n!} e^{-\frac{1}{2}x^2} L_n(x^2),$$

then $\phi(x)$ and $\psi(x)$ are Fourier sine-transforms of each other provided that n is a positive integer.

The following two lemmas are required:

LEMMA 1. *If n is a positive integer and $m > -\frac{1}{2}$, then*

$$T^n_m(x) - \frac{n+1}{2} T^{n+1}_m(x) + \frac{1.3.(n+1)(n+2)}{2.4} T^{n+2}_m(x) - \ldots = \frac{1}{\sqrt{x}} T^n_{m-\frac{1}{2}}(x).$$

In a recent paper* I have shown that

$$e^{-\frac{1}{2}x} x^m T^n_m(x) \doteqdot \frac{p(\frac{1}{2}-p)^n}{n!\,(\frac{1}{2}+p)^{m+n+1}}.$$

This gives

$$x^m T^n_m(x) \doteqdot \frac{(1-p)^n}{n!\,p^{m+n}} \tag{1}$$

by the help of the operational formula,

$$e^{-\alpha x} f(x) \doteqdot \frac{p}{p+\alpha} \phi(p+\alpha),$$

where

$$f(x) \doteqdot \phi(p),$$

Now consider the series

$$\frac{(1-p)^n}{n!\,p^{m+n}} \left\{1 - \frac{\frac{1}{2}}{1!} \frac{1-p}{p} + \frac{\frac{1}{2} \cdot \frac{3}{2}}{2!} \left(\frac{1-p}{p}\right)^2 - \ldots\right\} = \frac{1}{n!} \frac{(1-p)^n}{p^{m+n-\frac{1}{2}}}.$$

If we interpret it term by term, we at once obtain the above lemma.

* R. S. Varma, 'Some functions which are self-reciprocal in the Hankel transform': *Proc. London Math. Soc.* (2), 42 (1937), 9–17.

Since $$T^n_{\frac{1}{2}}(x) = \frac{2^{n+\frac{1}{2}}}{(2n+1)!\sqrt{\pi}} x^{-\frac{1}{2}} e^{\frac{1}{2}x} D_{2n+1}[\sqrt{(2x)}],$$

this gives, for $m = \frac{1}{2}$,

$$\frac{2^{n+1}\sqrt{(2\pi)}}{(2n+1)!}\left\{D_{2n+1}(x\sqrt{2}) - \frac{1}{2(2n+3)} D_{2n+3}(x\sqrt{2}) + \right.$$
$$\left. + \frac{1.3}{2.4.(2n+3)(2n+5)} D_{2n+5}(x\sqrt{2}) - \dots\right\} = 2\pi e^{-\frac{1}{2}x^2} T^n_0(x^2). \quad (2)$$

LEMMA 2. *If n is a positive integer,*

$$(-)^n i\{D^2_{-(n+1)}[i\sqrt{(2x)}] - D^2_{-(n+1)}[-i\sqrt{(2x)}]\}$$
$$= \frac{2^{n+1}\sqrt{(2\pi)}}{(2n+1)!}\left\{D_{2n+1}(2\sqrt{x}) + \frac{1}{2(2n+3)} D_{2n+3}(2\sqrt{x}) + \right.$$
$$\left. + \frac{1.3}{2.4.(2n+3)(2n+5)} D_{2n+5}(2\sqrt{x}) + \dots\right\}.$$

This has been proved by S. C. Mitra.*

To establish the above theorem, let us first restrict ourselves to even integral values of n. We then have, using (2),

$$\sqrt{\left(\frac{2}{\pi}\right)} \int\limits_0^\infty 2\pi e^{-\frac{1}{2}y^2} T^n_0(y^2) \sin xy \, dy$$
$$= \sqrt{\left(\frac{2}{\pi}\right)} \int\limits_0^\infty \frac{2^{n+1}\sqrt{(2\pi)}}{(2n+1)!}\left\{D_{2n+1}(y\sqrt{2}) - \frac{1}{2(2n+3)} D_{2n+3}(y\sqrt{2}) + \right.$$
$$\left. + \frac{1.3}{2.4.(2n+3)(2n+5)} D_{2n+5}(y\sqrt{2}) - \dots\right\} \sin xy \, dy.$$

Integrating term by term—a process obviously justifiable—and making use of the formula†

$$\sqrt{\left(\frac{2}{\pi}\right)} \int\limits_0^\infty D_{2m+1}(y\sqrt{2}) \sin xy \, dy = (-)^m D_{2m+1}(x\sqrt{2}),$$

* S. C. Mitra, 'The operational representation of $D_n(x)$ and

$$\{D^2_{-(n+1)}(ix) - D^2_{-(n+1)}(-ix)\},$$

Proc. Edinburgh Math. Soc. **4** (1934), 33–5.

† A. Milne, 'On the equation of the parabolic cylinder function': *Proc. Edinburgh Math. Soc.* (1), **32** (1914), 2–14. See also E. C. Titchmarsh, *Introduction to the theory of Fourier Integrals* (Oxford, 1937), 261.

we obtain

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty 2\pi e^{-\frac{1}{2}y^2}T_0^n(y^2)\sin xy\,dy$$
$$= \frac{2^{n+1}\sqrt{(2\pi)}}{(2n+1)!}\left\{D_{2n+1}(x\sqrt{2})+\frac{1}{2.(2n+3)}D_{2n+3}(x\sqrt{2})+\right.$$
$$\left.+\frac{1.3}{2.4.(2n+3)(2n+5)}D_{2n+5}(x\sqrt{2})+\ldots\right\}$$
$$= i\{D^2_{-(n+1)}(ix)-D^2_{-(n+1)}(-ix)\}$$
$$= \phi(x), \tag{3}$$

where we have used Lemma 2.

Proceeding in the above manner, it can be shown that

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \phi(y)\sin xy\,dy = 2\pi e^{-\frac{1}{2}x^2}T_0^n(x^2). \tag{4}$$

It is also easy to see that the results (3) and (4) remain true when n is an odd integer. Moreover, Sonine's polynomial is related to ordinary Laguerre polynomial by means of the relation

$$T_0^n(x) = \frac{(-1)^n}{n!}L_n(x).$$

This proves the theorem stated above.

SOME THEOREMS ON FUNCTIONS REGULAR IN AN ANGLE

By V. GANAPATHY IYER (*Madras*)

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1. I HAVE shown* recently that there exist sequences of points which resemble closely the lattice-points

$$z = m+in \quad (m,n = 0, \pm 1, \pm 2,...), \tag{1}$$

so far, at any rate, as the behaviour of integral functions at these points are concerned. Many writers† have considered the properties of functions regular in an angle in relation to the lattice-points in that angle. The main object of this paper is to examine whether the allied sequences mentioned above possess similar properties in relation to functions regular in an angle.

1.1. Let $z = re^{i\theta}$ and let $\Lambda(\alpha)$ denote the region

$$|\theta| \leqslant \alpha \quad (0 < \alpha \leqslant \pi), \tag{2}$$

where α is fixed in any particular discussion. Let $f(z)$ be a function regular in $\Lambda(\alpha)$ except perhaps at $z = 0$, where it is supposed to be continuous as $z \to 0$ from within $\Lambda(\alpha)$. We write

$$M\{r,f,\Lambda(\alpha)\} = \max|f(z)|$$

as z varies in the sector common to $\Lambda(\alpha)$ and the circle $|z| \leqslant r$. The order ρ and the type $\kappa\{f,\Lambda(\alpha)\}$ of $f(z)$ in $\Lambda(\alpha)$ are defined by the relations

$$\rho = \varlimsup_{r\to\infty} \frac{\operatorname{loglog} M\{r,f,\Lambda(\alpha)\}}{\log r}, \qquad \kappa\{f,\Lambda(\alpha)\} = \varlimsup_{r\to\infty} \frac{\log M\{r,f,\Lambda(\alpha)\}}{r^\rho}.$$

When $f(z)$ is an integral function we write $M(r,f) = M\{r,f,\Lambda(\pi)\}$ and $\kappa(f) = \kappa\{f,\Lambda(\pi)\}$. We shall denote by $C(\rho,d)$ the class of all integral functions of order ρ and type *less* than d, that is, all those functions $f(z)$ satisfying the condition

$$\varlimsup_{r\to\infty} \frac{\log M(r,f)}{r^\rho} < d,$$

where ρ and d are two finite positive numbers.

* *Trans. American Math. Soc.* 42 (1937), 358–65. This will be referred to as E in the sequel.

† M. L. Cartwright, *Proc. London Math. Soc.* (2), 43 (1937), 26–32. A. Pfluger, ibid. (2), 42 (1937), 305–15. J. M. Whittaker, *Proc. Edinburgh Math. Soc.* 2 (1930), 111–28; *Proc. London Math. Soc.* (2), 37 (1934), 383–401; *Interpolatory Function Theory* (Cambridge Math. Tracts, 33, 1935).

1.2. We now proceed to specify the kind of sequences to be considered in this paper. Let $[z_n]$ $(z_n = r_n e^{i\theta_n})$ be a sequence of distinct complex numbers such that

$$0 < r_1 \leqslant r_2 \leqslant r_3 \ldots \leqslant r_n \to \infty$$

as $n \to \infty$. For shortness we shall refer to the exponent of convergence of $[z_n]$ as its *order*. Let B be a positive constant and let $A_n(h)$ denote the circle with centre z_n and radius $B|z_n|^{-h}$. We denote by $A(h)$ the system of circles $\{A_n(h)\}$. Let ρ and d be two finite positive numbers. We shall say that the sequence $[z_n]$ belongs to *the class* $L(\rho, d)$ when there exists an integral function $g(z)$ of order ρ with simple zeros at the points of $[z_n]$ (and having no other zeros) satisfying the two following conditions:

$$\left.\begin{aligned} &\text{(i)}\ \lim_{n\to\infty} \frac{\log|g'(z_n)|}{|z_n|^\rho} = d, \\ &\text{(ii)}\ \frac{\log|g(z)|}{|z|^\rho} \to d \end{aligned}\right\} \tag{3}$$

as $|z| \to \infty$ outside the circles $A(h)$ for some positive B and some h greater than ρ.

1.3. The lattice-points (1) are the simplest known set of the type discussed in § 1.2. It belongs to the class $L(2, \frac{1}{2}\pi)$ and (ii) of (3) holds for any positive B and any h greater than 2. It is easily shown by using classical theorems* on integral functions that any sequence $[z_n]$ for which (ii) of (3) holds is of order ρ so that the sum of the diameters of the circles of $A(h)$ is finite. I have proved (E, Theorem 1) that, if $[z_n]$ belongs to $L(\rho, d)$, the set $[\theta_n]$ of the amplitudes of the points of $[z_n]$ is everywhere dense in $0 \leqslant \theta \leqslant 2\pi$, so that every angle however small contains an infinity of points of $[z_n]$. The sequence (E, Lemma 4)

$$\left[(2\rho d)^{-1/\rho} n^{2/\rho} e^{(2\nu\pi i)/n}\right] \quad \begin{pmatrix} n = 1, 2, 3, \ldots; \\ \nu = 0, 1, 2, \ldots, n-1 \end{pmatrix} \tag{4}$$

belongs to $L(\rho, d)$. The method used to prove that (4) belongs to $L(\rho, d)$ can be easily modified to prove the following lemma which gives a general class of sequences belonging to $L(\rho, d)$.

* Cf. my papers in *J. Indian Math. Soc.* (New Series), 2 (1936), 4–6 and 57–9.

LEMMA 1. *Let $l(x)$ and $m(x)$ be two differentiable functions of x in $0 < x < \infty$ tending to zero as $x \to \infty$. As $x \to \infty$, let*

$$xl'(x) \to 0 \quad \text{and} \quad x \log x\, m'(x) \to 0.$$

Let α, η, λ, μ, L, and M be positive constants. Let

$$[\lambda_n] \quad (0 < \lambda_1 < \lambda_2 \dots < \lambda_n \to \infty)$$

be a sequence defined by

$$\lambda_n = Ln^{\lambda}\{1+l(n)\}. \tag{5}$$

Let $[\mu_n]$ $(0 < \mu_1 \leqslant \mu_2 \leqslant \mu_3 \dots \leqslant \mu_n \to \infty)$ be a sequence of integers defined by

$$\mu_n = Mn^{\mu}\{1+m(n)\}. \tag{6}$$

Then the sequence

$$[\eta\lambda_n^{\alpha}\, e^{(2\nu\pi i)/\mu_n}] \quad \begin{pmatrix} n = 1, 2, 3, \dots; \\ \nu = 0, 1, 2, \dots, (\mu_n - 1) \end{pmatrix} \tag{7}$$

belongs to $L(\rho, d)$ where

$$\rho = \frac{\mu+1}{\lambda\alpha}, \qquad d = \frac{M\lambda\alpha}{\eta^{\rho}(\mu+1)^2 L^{(\mu+1)/\lambda}}. \tag{8}$$

1.4. Given ρ, d, L, M, λ, μ, the numbers α and η can be so determined from the relations (8) that (7) belongs to $L(\rho, d)$. The main properties of integral functions in relation to a sequence of $L(\rho, d)$ are contained in the following theorem.

THEOREM 1.* *Let $f(z)$ be an integral function of the class $C(\rho, d)$ [§ 1.1] and let $[z_n]$ be a sequence of $L(\rho, d)$. Then*

$$\kappa(f) = \overline{\lim_{r\to\infty}} \frac{\log M(r,f)}{r^{\rho}} = \overline{\lim_{n\to\infty}} \frac{\log|f(z_n)|}{|z_n|^{\rho}}.$$

Also, a function of $C(\rho, d)$ bounded at the points of a sequence of $L(\rho, d)$ must be a constant.

1.5. In this paper I propose to prove results similar to those of Theorem 1 for functions regular in an angle. The method used closely resembles the one I have adopted elsewhere† to derive the asymptotic behaviour of a function along a straight line when its behaviour at a sequence of points on the line is known. The special methods that have been used in the case of the lattice-points by the previous writers cannot be applied here, since these make use of the peculiar properties of lattice-points considered as complex integers.

* E, Theorem 2 and footnote on p. 365.

† *Quart. J. of Math.* (Oxford), 8 (1937), 131–41, Theorem 1.

In Theorems 2 and 3 below, the method used here has necessitated the assumption that the sides of the angle $\Lambda(\alpha)$ have no point in common with the circles of $A(h)$ except perhaps with a finite number of them. It is very likely that this assumption is quite unnecessary, since it does not appear to have any bearing on the remaining hypotheses and conclusions of these theorems. I show later (Theorem 4) that in the case of the lattice-points (1) and the sequences (4) and (7) this hypothesis can be dispensed with; in fact, I show that this is so in the case of a definite sub-class of $L(\rho, d)$. But I have not been able to get rid of this assumption in the general case of a sequence of $L(\rho, d)$.

2. The two following theorems constitute the main results of this paper.

THEOREM 2. *Let $f(z)$ be regular in $\Lambda(\alpha)$ and let $[z_n]$ be a sequence of $L(\rho, d)$. Let $A(h)$ be the system of circles in* (ii) *of* (3) *and let the following conditions hold:*

(i) *the sides $\theta = \pm\alpha$ of $\Lambda(\alpha)$ have no point in common with $A(h)$ except perhaps with a finite number of these circles;*

(ii) $\displaystyle \varliminf_{r\to\infty} \frac{\log M\{r, f, \Lambda(\alpha)\}}{r^\rho} < d;$

(iii) $\displaystyle \varlimsup_{r\to\infty} \frac{\log|f(re^{\pm i\alpha})|}{r^\rho} \leqslant \beta < d;$

(iv) $\displaystyle \varlimsup_{n\to\infty} \frac{\log|f(z_n)|}{|z_n|^\rho} \leqslant \beta < d \quad \{z_n \subset \Lambda(\alpha)\}.$

Then
$$\kappa\{f, \Lambda(\alpha)\} = \varlimsup_{r\to\infty} \frac{\log M\{r, f, \Lambda(\alpha)\}}{r^\rho} \leqslant \beta < d. \tag{9}$$

THEOREM 3. *Let $f(z)$ and $[z_n]$ be as in Theorem* 2. *Let the conditions* (i) *and* (ii) *of Theorem* 2 *hold. Let*

(iii)′ $|f(re^{\pm i\alpha})| \leqslant B;$

(iv)′ $|f(z_n)| \leqslant B_1$ *for* $z_n \subset \Lambda(\alpha).$

Then, for all z in $\Lambda(\alpha)$,
$$|f(z)| \leqslant B. \tag{10}$$

2.1. We shall first establish a few lemmas. Lemma 2 has been proved in E, Lemma 2.

LEMMA 2. *The function $g(z)$ in the definition of the class $L(\rho, d)$* [§ 1.2 (3)] *is of order ρ and type d.*

LEMMA 3. *Let $\lambda > 0$. Then any ring $R-\lambda \leqslant |z| \leqslant R$ contains a circle not cutting any circle of the system $A(h)$ provided that $h > \rho$ and $R \geqslant R_0 = R_0(\lambda)$.*

Proof. This follows from the fact that $\sum |z_n|^{-h}$ converges when $h > \rho$.

LEMMA 4. *Let the sides of $\Lambda(\alpha)$ have no point in common with $A(h)$. Let the remaining conditions of Theorem 2 hold. Let $\beta < \gamma < d$ and*

$$H_m(z) = z^m \frac{f(z)}{g(z)} - \sum_{z_n \subset \Lambda(\alpha)} z_n^m \frac{f(z_n)}{g'(z_n)} \frac{1}{z-z_n}, \tag{11}$$

where $m \geqslant 0$ is an integer and $g(z)$ is the function in the relation (3). *Then, for all z in $\Lambda(\alpha)$,*

$$|H_m(z)| \leqslant A(f,\gamma)A(g,\gamma)\left\{\sum_{z_n \subset \Lambda(\alpha)} |z_n|^{m+h} \exp[-(d-\gamma)|z_n|^\rho] + \left[\frac{m}{\rho e(d-\gamma)}\right]^{m/\rho}\right\}, \tag{12}$$

where $A(f,\gamma)$ is a constant depending on the function $f(z)$ and the number γ.*

Proof. Since $\theta = \pm\alpha$ have no point in common with $A(h)$, we have, for z on $\theta = \pm\alpha$,

$$\left|\sum_{z_n \subset \Lambda(\alpha)} \frac{z_n^m f(z_n)}{g'(z_n)} \frac{1}{z-z_n}\right| \leqslant A(f,\gamma)A(g,\gamma) \sum_{z_n \subset \Lambda(\alpha)} |z_n|^{m+h} \exp[-(d-\gamma)|z_n|^\rho], \tag{13}$$

in virtue of (i) of (3) and the condition (iv) of Theorem 2. By (ii) of (3) and the condition (iii) of Theorem 2 we get, for z on $\theta = \pm\alpha$,

$$\left|z^m \frac{f(z)}{g(z)}\right| \leqslant A(f,\gamma)A(g,\gamma)\left\{\max_{0 \leqslant r < \infty} r^m \exp[-(d-\gamma)r^\rho]\right\}$$
$$\leqslant A(f,\gamma)A(g,\gamma)\left[\frac{m}{\rho e(d-\gamma)}\right]^{m/\rho}. \tag{14}$$

* Throughout the rest of this paper an expression of the form

$$A(f, g, \ldots, \gamma, \lambda, \ldots)$$

will denote a finite positive constant depending on the functions $f, g, \ldots$ and the numbers $\gamma, \lambda, \ldots$ which need not have the same value at the different places where it occurs. It is supposed that the number B occurring in the radius of the circles of $A(h)$ is unity; otherwise B^{-1} will occur as factor in (12) and this does not affect the arguments in § 2.2.

By (ii) of Theorem 2 there is a δ $(0 < \delta < d)$ and a sequence

$$[R_n] \quad (0 < R_1 < R_2 ... < R_n \to \infty),$$

such that
$$M\{R_n, f, \Lambda(\alpha)\} \leqslant \exp[(d-\delta)R_n^\rho]. \tag{15}$$

By Lemma 2 there is a circle in the ring $R_n - 1 \leqslant |z| \leqslant R_n$ which does not cut any circle of $A(h)$. If Γ_n denotes the arc of this circle in $\Lambda(\alpha)$,
$$|f(z)| \leqslant \exp[(d-\delta)R_n^\rho] \tag{16}$$
for z on Γ_n in virtue of (15). Since Γ_n is outside the circles $A(h)$, we can use (ii) of (3) and in conjunction with (16) conclude that

$$\left|z^m \frac{g(z)}{f(z)}\right| \to 0 \quad \text{as} \quad |z| \to \infty \text{ along the arcs } \Gamma_n. \tag{16'}$$

The relations (13), (14), (16′) give (12) in conjunction with the maximum-modulus principle.

2.2. *Proof of Theorem* 2. If we omit a finite number of points from a sequence of $L(\rho, d)$, the rest still belongs to $L(\rho, d)$. Therefore we can suppose without loss of generality in the condition (i) of Theorem 2 that no circle of $A(h)$ cuts $\theta = \pm\alpha$. We can now apply Lemma 4, and so the relation (12) is valid for any γ such that $\beta < \gamma < d$. Let $0 < \delta < d-\beta$, and choose λ so that $d\lambda^\rho = d-\beta-\delta$ (if $\beta = -\infty$ we can choose λ to be any large positive number). Let

$$\chi(z) = c_0 + c_1 z + c_2 z^2 + ...$$

be any integral function of order ρ and type not exceeding $d\lambda^\rho$. Then, by a classical theorem on integral functions,

$$\overline{\lim_{m\to\infty}}\, m^{1/\rho}|c_m|^{1/m} \leqslant (d\lambda^\rho e\rho)^{1/\rho}. \tag{17}$$

Take $\gamma = \beta + \frac{1}{2}\delta = d - d\lambda^\rho - \frac{1}{2}\delta$ in (12). Then we get in virtue of (i) of (3) and (17)

$$\begin{aligned}
&\left|\frac{f(z)\chi(z)}{g(z)} - \sum_{z_n \subset \Lambda(\alpha)} \frac{f(z_n)\chi(z_n)}{g'(z_n)} \frac{1}{z-z_n}\right| \\
&\leqslant A(f,\gamma)A(g,\gamma)\left\{\sum_{z_n \subset \Lambda(\alpha)} |z_n|^h \exp[-(d\lambda^\rho + \tfrac{1}{2}\delta)|z_n|^\rho]\left(\sum_{m=1}^{\infty} |c_m||z_n|^m\right)\right. \\
&\qquad\qquad\qquad\qquad \left. + \sum_{m=1}^{\infty}\left[\frac{|c_m|^{\rho/m} m}{\rho e(d\lambda^\rho + \frac{1}{2}\delta)}\right]^{m/\rho}\right\} \\
&\leqslant A(f,\gamma)A(g,\gamma)A(\chi,\rho,d,\lambda,\delta,h) \\
&\leqslant A(f,g,\chi,\rho,d,\beta,\delta,h).
\end{aligned} \tag{18}$$

By Lemma 2 we can take $\chi(z) = g(\lambda z)$ in (18). Then we get for z in $\Lambda(\alpha)$ but outside the circles $A(h)$,

$$|f(z)| \leqslant \left|\frac{g(z)}{g(\lambda z)}\right| A(f,g,\rho,d,\beta,\delta,h), \tag{19}$$

in virtue of (i) of (3) and the choice of λ. Let $A_\lambda(h)$ denote the system of circles round the zeros of $g(\lambda z)$ similar to $A(h)$. Then by (ii) of (3) and (19) we get that

$$\overline{\lim} \frac{\log|f(z)|}{|z|^\rho} \leqslant d - d\lambda^\rho = \beta + \tfrac{1}{2}\delta, \tag{20}$$

as $|z| \to \infty$ outside the circles of $A(h)$ and $A_\lambda(h)$. Let $\epsilon > 0$. We get from (20) that for any z outside these circles

$$|f(z)| \leqslant \exp[(\beta + \tfrac{1}{2}\delta + \epsilon)|z|^\rho] \tag{21}$$

provided that $|z| \geqslant r_0 = r_0(\epsilon)$. Now the system of circles $A(h)$ and $A_\lambda(h)$ together determine a sequence of non-overlapping domains $D_1, D_2, \ldots, D_3, \ldots$. The diameter of any D_n does not exceed the finite number

$$P = 2 \sum_{z_n \subset \Lambda(\alpha)} B|z_n|^{-h}(1+\lambda^{-h}).$$

Since (21) holds on the boundary of D_n for all $n \geqslant n_0$, and since the diameter of D_n does not exceed P, it follows from the maximum-modulus principle that, for z inside any D_n,

$$|f(z)| \leqslant \exp[(\beta + \tfrac{1}{2}\delta + 2\epsilon)|z|^\rho] \tag{22}$$

provided $n \geqslant n_1 = n_1(\epsilon)$. From (21) and (22) we conclude that

$$\overline{\lim} \frac{\log|f(z)|}{|z|^\rho} \leqslant \beta + \tfrac{1}{2}\delta, \tag{23}$$

as $|z| \to \infty$ in $\Lambda(\alpha)$. Since δ is as small as we please in (23), we get the relation (9). This proves Theorem 2.

2.3. *Proof of Theorem* 3. As in the proof of Theorem 2 we suppose that $\theta = \pm\alpha$ have no point in common with $A(h)$. Theorem 2 in conjunction with (iii)′ and (iv)′ of Theorem 3 gives

$$\kappa\{f, \Lambda(\alpha)\} = \overline{\lim_{r \to \infty}} \frac{\log M\{r, f, \Lambda(\alpha)\}}{r^\rho} \leqslant 0. \tag{24}$$

Hence (24) holds when $f(z)$ is replaced by $[f(z)]^p$, where $p \geqslant 0$ is any integer. Proceeding as in the proof of Lemma 4 and using (iii)′ and (iv)′ of the present theorem we get, for all z in $\Lambda(\alpha)$,

$$\left|\frac{[f(z)]^p}{g(z)} - \sum_{z_n \subset \Lambda(\alpha)} \frac{[f(z_n)]^p}{g'(z_n)} \frac{1}{z - z_n}\right| \leqslant A(g,d)B^p + A(g,d,h)B_1^p. \tag{25}$$

From (25) we get, for z in $\Lambda(\alpha)$ but outside $A(h)$,

$$\left|\frac{[f(z)]^p}{g(z)}\right| \leqslant A(g,d)B^p+2A(g,d,h)B_1^p,$$

from which we get, for any fixed z in $\Lambda(\alpha)$ but outside $A(h)$,

$$\begin{aligned}|f(z)| &\leqslant \overline{\lim_{p\to\infty}}\, |g(z)|^{1/p}[A(g,d)B^p+2A(g,d,h)B_1^p]^{1/p} \\ &\leqslant \max(B,B_1). \qquad (26)\end{aligned}$$

Using the maximum-modulus principle as in the last stages of the proof of Theorem 2, we find that (26) holds throughout $\Lambda(\alpha)$. Hence $f(z)$ is bounded in $\Lambda(\alpha)$ while $|f(re^{\pm i\alpha})| \leqslant B$, so that by applying a well-known theorem of Phragmén and Lindelöf* we find that (10) holds for all z in $\Lambda(\alpha)$.

3. We shall now show that in certain cases the condition (i) of Theorems 2 and 3 can be dispensed with. For this purpose we require a preliminary result. The circles $A(h)$ in (ii) of (3) determine a sequence of non-overlapping domains

$$[D_1, D_2,\ldots, D_n,\ldots] = [D_n] = D(h).$$

Let $D_1(h)$ be a sub-sequence of domains selected from those of $D(h)$. We shall prove the following

LEMMA 5. *Let $[z_n]$ be a sequence of $L(\rho,d)$. Let those points of $[z_n]$ lying in $D_1(h)$ be a sequence of order less than ρ. Then the sequence of points of $[z_n]$ not lying in $D_1(h)$ belongs to $L(\rho,d)$.*

Proof. Let the points of $[z_n]$ lying in $D_1(h)$ be of order $\rho_1 < \rho$ and let $\sigma(z)$ be the canonical product with simple zeros at these points. Then, if $\rho_1 < \rho_2 < \rho$, we have†

$$\log|\sigma(z)| = O(|z|^{\rho_2}), \qquad (27)$$

as $|z| \to \infty$ outside $D_1(h)$. Let $D_2(h)$ denote those domains of $D(h)$ not belonging to $D_1(h)$ and let

$$G(z) = \frac{g(z)}{\sigma(z)}.$$

Then $G(z)$ is an integral function having simple zeros at those

* G. Valiron, *Lectures on Integral Functions*, p. 125.

† Here we use the fact that, *if $g(z)$ be a function of order ρ and $A(h)$ denote the circles round the zeros of $g(z)$ as in* § 1.2, *we have*

$$\log|g(z)| = O(|z|^{\rho'}) \quad (\rho' > \rho)$$

as $|z| \to \infty$ outside $A(h)$ for any finite h. This result can be obtained by arguments similar to those used in the usual proof of Hadamard's theorem on the minimum modulus of integral functions.

points of $[z_n]$ lying in $D_2(h)$ and having no other zeros. In virtue of (27) and the fact that the domains of $D(h)$ are non-overlapping we get, for $z_n \subset D_2(h)$,

$$\log|G'(z_n)| = \log|g'(z_n)| + O(|z_n|^{\rho_2}), \tag{28}$$

so that (i) of (3) holds for the points of $[z_n]$ in $D_2(h)$. Also, by (ii) of (3) and the fact that $D_1(h) \subset D(h)$, we get

$$\log|G(z)| = \log|g(z)| + O(|z|^{\rho_2}) \tag{29}$$

for z outside $A(h) = D(h)$. Hence

$$\lim \frac{\log|G(z)|}{|z|^{\rho}} = d, \tag{30}$$

as $|z| \to \infty$ outside $A(h) = D(h)$. Since $G(z)$ does not vanish in $D_1(h)$, $\log|G(z)|$ is harmonic therein. Using the fact that a function harmonic in a domain D cannot attain its maximum or minimum in the interior of D and arguing as in the last stages of the proof of Theorem 2 we can conclude that (30) holds as $|z| \to \infty$ in $D_1(h)$ as well. Hence (30) holds as $|z| \to \infty$ outside $D_2(h)$. This implies that (ii) of (3) holds for points of $[z_n]$ lying in $D_2(h)$. This proves the lemma.

3.1. Now let $[z_n]$ be a sequence of $L(\rho, d)$. Let $A(h)$, $D(h)$ be as in § 3. Let $[z_n, \theta_0, D(h)]$ denote those points of $[z_n]$ that lie in such domains of $D(h)$ as have a point in common with the line $\theta = \theta_0$. *If for every θ in $0 \leqslant \theta \leqslant 2\pi$ the sequence $[z_n, \theta, D(h)]$ is of order less than ρ, we shall say that $[z_n]$ belongs to the class $L_1(\rho, d)$.* It is easy to see that the lattice-points (1) and the sequences (4) and (7) belong to $L_1(\rho, d)$. Using Lemma 5 we see that a sequence $[z_n]$ of $L_1(\rho, d)$ still belongs to the class when the points of $[z_n, \alpha, D(h)]$ and $[z_n, -\alpha, D(h)]$ are omitted from it. Hence in the case of such sequences the conditions (i) of Theorems 2 and 3 can be omitted and the theorems still remain true. So we can state

THEOREM 4. *Let $[z_n]$ be a sequence of $L_1(\rho, d)$. Then Theorems* 2 *and* 3 *hold even when conditions* (i) *of these theorems are dispensed with. In particular this is true in the case of the sequences* (1), (4), (7).

3.2. It is a significant feature of the results of this paper that the magnitude of the angle α of $\Lambda(\alpha)$ does not affect the truth of the theorems proved here. If $\alpha < \pi/2\rho$, the conditions (ii) and (iii) of Theorem 2 and (ii) and (iii)′ of Theorem 3 are sufficient to ensure that (9) and (10) hold in the respective cases. But, if $\alpha \geqslant \pi/2\rho$, this is no longer true and in this case the conditions (iv) and (iv)′ come

into play. In this connexion it is interesting to compare the following result due to Miss Cartwright.*

THEOREM 5. *Let $f(z)$ be regular in $\Lambda(\alpha)$. Let the following conditions hold*:

(i) $\overline{\lim\limits_{r\to\infty}} \dfrac{\log|f(re^{i\theta})|}{r^2} \leqslant \lambda < \frac{1}{2}\pi$ *for* $\alpha-\delta \leqslant |\theta| \leqslant \alpha$ *for some positive* δ *and* $\lambda < \frac{1}{2}\pi\sin^2\alpha$ *if* $\alpha < \frac{1}{4}\pi$;

(ii) $\underline{\lim\limits_{r\to\infty}} \dfrac{\log M\{r,f,\Lambda(\alpha)\}}{r^2} < \frac{1}{2}\pi$;

(iii) $f(m+in) = O(1)$ *for all those lattice-points lying in* $\Lambda(\alpha)$. *Let* $\beta < \alpha$ *be such that* $\sin^2(\alpha-\beta) > 2\lambda/\pi$. *Then* $f(z) = O(1)$ *for* z *in* $\Lambda(\beta)$.

3.3. The above theorem states that under the conditions (i), (ii), and (iii) of the theorem there are angles inside $\Lambda(\alpha)$ in which $f(z)$ is bounded. Since the hypotheses are with reference to $\Lambda(\alpha)$ and the result proved is for an angle $\Lambda(\beta)$ ($\beta < \alpha$), a condition similar to (i) of Theorem 3 is fulfilled effectively. In Theorem 3 condition (iii)$'$ replaces condition (i) of Theorem 5, which is less stringent. But, as shown in Theorem 4, the condition (i) of Theorem 3 on the boundary of $\Lambda(\alpha)$ is unnecessary in the case of lattice-points. It is very probable that a result of the type in Theorem 5 holds for any sequence of $L(\rho, d)$, though the methods of this paper seem insufficient to prove it. But Theorem 2 of this paper enables us to replace (i) of Theorem 5 by *the less restrictive condition*

$$\overline{\lim_{r\to\infty}} \frac{\log|f(re^{\pm i\alpha})|}{r^2} \leqslant \lambda < \tfrac{1}{2}\pi, \tag{31}$$

since (31) in conjunction with (ii) and (iii) of Theorem 5 implies that

$$\overline{\lim_{r\to\infty}} \frac{\log M\{r,f,\Lambda(\alpha)\}}{r^2} \leqslant \lambda < \tfrac{1}{2}\pi$$

and the last relation contains, *a fortiori*, the condition (i) of Theorem 5.

[*Added 21 July 1938*]

A step has been omitted in the proof of Lemma 4. In addition to (16$'$), it is necessary to show that the expression on the left of (13) tends to zero as $|z| \to \infty$ along the arcs Γ_n. But it is easy to see that this expression tends to zero as $|z| \to \infty$ outside $A(h)$ by considering the two sums corresponding to $|z_n| \leqslant (1-\lambda)|z|$ and $|z_n| > (1-\lambda)|z|$ where $0 < \lambda < 1$. *A fortiori* the same is true as $|z| \to \infty$ along Γ_n.

* See M. L. Cartwright, loc. cit.

ON DIVISOR PROBLEMS

By E. C. TITCHMARSH (*Oxford*)

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1. Let $d_k(n)$ denote the number of expressions of n as a product of k factors, and let
$$D_k(x) = \sum_{n \leqslant x} d_k(n).$$

It is known that
$$D_k(x) = (a_{0,k} + a_{1,k} \log x + \ldots + a_{k-1,k} \log^{k-1} x)x + \Delta_k(x)$$
where $\Delta_k(x) = o(x)$. We may define the order α_k of $\Delta_k(x)$ as the lower bound of numbers λ such that $\Delta_k(x) = O(x^\lambda)$. The problem of determining α_k is notoriously difficult.* It was proved by Hardy† that $\alpha_k \geqslant \frac{1}{2}(k-1)/k$, but the exact value of α_k is not known for any k.

An easier problem is that of determining β_k, the average order of $\Delta_k(x)$, defined as the lower bound of numbers λ such that
$$\frac{1}{x} \int\limits_1^x \Delta_k^2(y)\, dy = O(x^{2\lambda}).$$
Clearly $\beta_k \leqslant \alpha_k$. It was proved by Hardy† that $\beta_2 \leqslant \frac{1}{4}$; by Cramer‡ that $\beta_k \leqslant 1-2/k$ for $k \geqslant 3$; and by Hardy and Littlewood§ that, on the unproved Lindelöf hypothesis,
$$\beta_k = \tfrac{1}{2}(k-1)/k$$
for every k.

All these results depend more or less on the Riemann zeta-function. The connexion is given by the formula
$$\{\zeta(s)\}^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} \quad (\sigma > 1).$$
Apparently no one has pointed out explicitly the exact relation between the β_k and the theory of the zeta-function. It is, however, quite simple. Let σ_k be the lower bound of values of σ for which
$$\int\limits_1^T |\zeta(\sigma+it)|^{2k}\, dt = O(T).$$
Then, for each integer k, a necessary and sufficient condition that $\beta_k = \frac{1}{2}(k-1)/k$ is that $\sigma_k \leqslant \frac{1}{2}(k+1)/k$.

* See Hardy and Littlewood (**5**).

† Hardy (**3**), (**4**).

‡ Cramer (**1**).

§ Hardy and Littlewood (**6**).

I also prove that $\beta_2 = \frac{1}{4}$, $\beta_3 = \frac{1}{3}$, that $\frac{3}{8} \leqslant \beta_4 \leqslant \frac{3}{7}$, and that $\beta_k \geqslant \frac{1}{2}(k-1)/k$ for every k.

2. Perron's formula for the sum of the coefficients in a Dirichlet series gives

$$D_k(x) = \frac{1}{2\pi i} \lim_{T\to\infty} \int\limits_{c-iT}^{c+iT} \frac{\zeta^k(s)}{s} x^s\,ds \quad (c > 1). \tag{2.1}$$

Applying Cauchy's theorem to the rectangle $\gamma - iT$, $c-iT$, $c+iT$, $\gamma+iT$, where γ is less than but sufficiently near to 1, and allowing for the residue at $s = 1$, we obtain

$$\Delta_k(x) = \frac{1}{2\pi i} \lim_{T\to\infty} \int\limits_{\gamma-iT}^{\gamma+iT} \frac{\zeta^k(s)}{s} x^s\,ds. \tag{2.2}$$

Let γ_k be the lower bound of positive numbers σ for which

$$\int\limits_{-\infty}^{\infty} \frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2}\,dt < \infty. \tag{2.3}$$

Then (2.2) holds for $\gamma_k < \gamma < 1$. Since $\zeta(s) = O(|t|^A)$ uniformly for $\sigma \geqslant 0$, it follows from a general theorem on mean-values of analytic functions* that the left-hand side of (2.3) is a bounded function of σ for $\gamma_k + \epsilon \leqslant \sigma \leqslant 1-\epsilon$ $(\epsilon > 0)$. Hence† $\zeta^k(s)/s \to 0$ uniformly as $t \to \pm\infty$ inside any strip $\gamma_k + \epsilon \leqslant \sigma \leqslant 1-\epsilon$, where $\epsilon > 0$. Hence, if we integrate the integrand of (2.2) round the rectangle $\gamma'-iT$, $\gamma-iT$, $\gamma+iT$, $\gamma'+iT$, where $\gamma_k < \gamma' < \gamma < 1$, and make $T \to \infty$, we obtain the same result with γ' instead of γ.

If we replace x by $1/x$, (2.2) expresses the relation between the Mellin transforms

$$f(x) = \Delta_k(1/x), \qquad \mathfrak{F}(s) = \zeta^k(s)/s,$$

the relevant integrals holding also in the mean-square sense. Hence Parseval's formula for Mellin transforms‡ gives

$$\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2}\,dt = \int\limits_0^{\infty} \Delta_k^2\left(\frac{1}{x}\right) x^{2\sigma-1}\,dx = \int\limits_0^{\infty} \Delta_k^2(x) x^{-2\sigma-1}\,dx, \tag{2.4}$$

provided that $\gamma_k < \sigma < 1$.

* Hardy, Ingham, and Pólya (**7**), Theorem 7.

† See, e.g., Titchmarsh (**9**), § 5.4, lemma.

‡ See Titchmarsh (**9**), Theorem 71.

It follows that, if $\gamma_k < \sigma < 1$,

$$\int_{\frac{1}{2}X}^{X} \Delta_k^2(x)x^{-2\sigma-1}\,dx < C = C(k,\sigma),$$

$$\int_{\frac{1}{2}X}^{X} \Delta_k^2(x)\,dx < CX^{2\sigma+1},$$

and, replacing X by $\frac{1}{2}X$, $\frac{1}{4}X$,... and adding,

$$\int_{1}^{X} \Delta_k^2(x)\,dx < CX^{2\sigma+1}.$$

Hence $\beta_k \leqslant \sigma$ for all such σ, i.e. $\beta_k \leqslant \gamma_k$.

The inverse Mellin formula is

$$\frac{\zeta^k(s)}{s} = \int_0^\infty \Delta_k\left(\frac{1}{x}\right)x^{s-1}\,dx = \int_0^\infty \Delta_k(x)x^{-s-1}\,dx. \tag{2.5}$$

The right-hand side exists primarily in the mean-square sense, for $\gamma_k < \sigma < 1$. But actually the right-hand side is uniformly convergent in any region interior to the strip $\beta_k < \sigma < 1$. For

$$\int_{\frac{1}{2}X}^{X} |\Delta_k(x)|x^{-\sigma-1}\,dx \leqslant \left\{\int_{\frac{1}{2}X}^{X} \Delta_k^2(x)\,dx \int_{\frac{1}{2}X}^{X} x^{-2\sigma-2}\,dx\right\}^{\frac{1}{2}}$$
$$= \{O(X^{2\beta_k+1+\epsilon})O(X^{-2\sigma-1})\}^{\frac{1}{2}} = O(X^{\beta_k-\sigma+\frac{1}{2}\epsilon}),$$

and on putting $X = 2, 4, 8,...$ and adding, we obtain

$$\int_1^\infty |\Delta_k(x)|x^{-\sigma-1}\,dx < C \quad (\beta_k < \sigma < 1).$$

It follows that the right-hand side of (2.5) represents an analytic function, regular for $\beta_k < \sigma < 1$. The formula therefore holds by analytic continuation throughout this strip. Also, by the argument just given, the right-hand side of (2.4) is finite for $\beta_k < \sigma < 1$. Hence, by the Plancherel theory of Mellin transforms, so is the left-hand side, and the formula holds. Hence $\gamma_k \leqslant \beta_k$, and so in fact $\gamma_k = \beta_k$.

3. We can now prove that $\beta_k \geqslant \frac{1}{2}(k-1)/k$. If $\frac{1}{2} < \sigma < 1$ we have

$$CT < \int_{\frac{1}{2}T}^{T} |\zeta(\sigma+it)|^2\,dt \leqslant \left(\int_{\frac{1}{2}T}^{T} |\zeta(\sigma+it)|^{2k}\,dt\right)^{1/(2k)}\left(\int_{\frac{1}{2}T}^{T} dt\right)^{1-1/(2k)},$$

whence
$$\int\limits_{\frac{1}{2}T}^{T} |\zeta(\sigma+it)|^{2k}\,dt > C'T.$$

Hence, if $0 < \sigma < \frac{1}{2}$, $T > 1$,

$$\int\limits_{-\infty}^{\infty} \frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2}\,dt > \int\limits_{\frac{1}{2}T}^{T} \frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2}\,dt > \frac{C}{T^2}\int\limits_{\frac{1}{2}T}^{T} |\zeta(\sigma+it)|^{2k}\,dt$$

$$> CT^{k(1-2\sigma)-2}\int\limits_{\frac{1}{2}T}^{T} |\zeta(1-\sigma-it)|^{2k}\,dt > CT^{k(1-2\sigma)-1}.$$

This can be made as large as we please by choice of T if $\sigma < \frac{1}{2}(k-1)/k$. Hence $\gamma_k \geqslant \frac{1}{2}(k-1)/k$, i.e. $\beta_k \geqslant \frac{1}{2}(k-1)/k$.

To prove the theorem stated in § 1, suppose first that $\sigma_k \leqslant \frac{1}{2}(k+1)/k$. Then, by the functional equation,

$$\int\limits_{1}^{T} |\zeta(\sigma+it)|^{2k}\,dt = O\left\{T^{k(1-2\sigma)}\int\limits_{1}^{T} |\zeta(1-\sigma-it)|^{2k}\,dt\right\} = O(T^{k(1-2\sigma)+1})$$

for $\sigma < \frac{1}{2}(k-1)/k$. It follows from the convexity of mean values of analytic functions* that for $\frac{1}{2}(k-1-\epsilon)/k < \sigma < \frac{1}{2}(k+1+\epsilon)/k$

$$\int\limits_{1}^{T} |\zeta(\sigma+it)|^{2k}\,dt = O\left(T^{1+\left(\frac{1}{2}+\frac{1}{2k}+\frac{\epsilon}{2k}-\sigma\right)k}\right).$$

This is of the form $O(T^{2-\delta})$ $(\delta > 0)$, if $\sigma > \frac{1}{2}(k-1+\epsilon)/k$. Then

$$\int\limits_{\frac{1}{2}T}^{T} \frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2}\,dt = O(T^{-\delta}),$$

and it follows that the left-hand side of (2.4) is finite. Hence

$$\gamma_k \leqslant \tfrac{1}{2}(k-1)/k.$$

Hence $\gamma_k = \beta_k = \frac{1}{2}(k-1)/k$.

On the other hand, if $\beta_k = \frac{1}{2}(k-1)/k$, it follows from (2.4) that

$$\int\limits_{1}^{T} |\zeta(\sigma+it)|^{2k}\,dt = O(T^2)$$

for $\sigma > \frac{1}{2}(k-1)/k$. Hence, by the functional equation,

$$\int\limits_{1}^{T} |\zeta(\sigma+it)|^{2k}\,dt = O(T^{k(1-2\sigma)-2})$$

* Hardy, Ingham, and Pólya (7), Theorem 6.

for $\sigma < \frac{1}{2}(k+1)/k$. Hence, by the convexity theorem, the left-hand side is $O(T^{1+\epsilon})$ for $\sigma = \frac{1}{2}(k+1)/k$, and so, by an argument due to Carlson,* is $O(T)$ for $\sigma > \frac{1}{2}(k+1)/k$. Hence $\sigma_k \leqslant \frac{1}{2}(k+1)/k$.

4. It is known† that $\sigma_k \leqslant 1-1/k$. Since $1-1/k \leqslant \frac{1}{2}(k+1)/k$ for $k \leqslant 3$, it follows that $\beta_2 = \frac{1}{4}$, $\beta_3 = \frac{1}{3}$.

The available material is not quite sufficient to determine β_4. We have $\beta_4 \geqslant \frac{3}{8}$. To obtain an upper bound for it, we observe that, since $\zeta(\frac{1}{2}+it) = O(t^{\frac{1}{6}+\epsilon})$,

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^8\,dt = O\left(T^{\frac{2}{3}+\epsilon}\int_1^T |\zeta(\tfrac{1}{2}+it)|^4\,dt\right) = O(T^{\frac{5}{3}+\epsilon}).$$

Also, since $\sigma_4 \leqslant \frac{7}{10}$,‡

$$\int_1^T |\zeta(\tfrac{3}{10}+it)|^8\,dt = O\left(T^{8/5}\int_1^T |\zeta(\tfrac{7}{10}-it)|^8\,dt\right) = O(T^{13/5+\epsilon}).$$

Hence by the convexity theorem

$$\int_1^T |\zeta(\sigma+it)|^8\,dt = O(T^{4-\frac{14}{3}\sigma+\epsilon}) \quad (\tfrac{3}{10} < \sigma < \tfrac{1}{2}).$$

This is of the form $O(T^{2-\delta})$ if $\sigma > \frac{3}{7}$, and it follows as before that $\beta_4 \leqslant \frac{3}{7}$. A slightly better result could be obtained by using more refined results about $\zeta(\frac{1}{2}+it)$.

* See Titchmarsh (8), § 2.42.

† Titchmarsh (8), § 2.51.

‡ Davenport (2).

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ELECTRICAL NOTES

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XI. MAGNETRON OSCILLATIONS*

[Received 11 March 1938]

12. Calculation of current for the first orbit

The convection current due to a single charge e is $de_2/c\,dt$, from (14), where (apart from a constant)

$$\pi e_2/e = \text{imaginary part of } [-\log(i-z/b)+\log(-i-z/b)] \tag{36}$$

and

$$z/b = \text{cm}\tfrac{1}{2}(\omega t-\omega t_1)\exp i[\theta_1+\tfrac{1}{2}(\omega t-\omega t_1)]. \tag{37}$$

The charge grazing the anode, in the steady motion, between polar angles θ_1 and $\theta_1+d\theta_1$ and between times t_1 and t_1+dt_1 is

$$-cI\,d\theta_1\,dt_1/2\pi;$$

and, as in Note VIII,† the current due to all the charges emitted so as to graze the anode between θ_1 and $\theta_1+d\theta_1$ is

$$I\,d\theta_1(f_2-f_1+f_4-f_3+...)/2\pi, \tag{38}$$

where f stands for the right-hand side of (36) and the electrons between the electrodes, beginning with those which left first, extend at time t from z_1 to z_2, from z_3 to z_4, and so on.

Let the period of the oscillations be the same as the time of one orbit, let $b/a = 500$ or $\nu/\omega = 0{\cdot}903$ as before, and let the range of νt_1 for anode capture be from α to $\pi+\alpha$, where α is known for every value of θ_1. Remembering that an electron caught by the anode is present in the field when $-\pi < \nu t-\nu t_1 < 0$ and that one not so caught is present when $-\pi < \nu t-\nu t_1 < \pi$, it is easy to see that electrons are present when $\nu t < \nu t_1 < 2\pi+\alpha$ if $\alpha < \nu t < \pi+\alpha$, and when $\pi+\alpha < \nu t_1 < \pi+\nu t$ if $\pi+\alpha < \nu t < 2\pi+\alpha$. That is, there are only two terms in equation (37), such that

$$\frac{z_1}{b} = \exp i\theta_1, \qquad \frac{z_2}{b} = \text{cm}\frac{\nu t-\alpha}{1{\cdot}806}\exp i\left[\theta_1+\frac{\nu t-\alpha}{1{\cdot}806}\right]$$

when $\alpha < \nu t < \pi+\alpha$, and

$$\frac{z_1}{b} = \text{cm}\frac{\nu t-\alpha-\pi}{1{\cdot}806}\exp i\left[\theta_1+\frac{\nu t-\alpha-\pi}{1{\cdot}806}\right], \qquad \frac{z_2}{b} = 0$$

* This note is a continuation of Notes IX and X, *Quart. J. of Math.* (Oxford), 7 (1936), 241, and 8 (1937), 75.

† *Quart. J. of Math.* (Oxford), 7 (1936), 211.

when $\pi+\alpha < \nu t < 2\pi+\alpha$. Let θ_1 lie at first between 0 and π. Then when $\alpha < \nu t < \pi+\alpha$, f_1 is $\frac{1}{2}$ or $\frac{3}{2}$ according as $\theta_1 <$ or $> \frac{1}{2}\pi$. If θ_1 is increased by π and νt is unaltered, f_2-f_1 is replaced by f_2-1 when $\alpha < \nu t < \pi+\alpha$ and by $\frac{1}{2}-f_1$ or $\frac{3}{2}-f_1$ when $\pi+\alpha < \nu t < 2\pi+\alpha$, according as $\theta_1 <$ or $> \frac{1}{2}\pi$. The calculations, therefore, need only be carried out in full for $0 < \theta_1 < \pi$; and as $\theta_1 = \frac{1}{2}\pi$ has to serve for a range of values round $\frac{1}{2}\pi$, the fractions $\frac{1}{2}$ and $\frac{3}{2}$ are then replaced by 1.

The table on pp. 223–4 gives the values of $1000(f_2-f_1)$ for $\theta_1 = 0°$ to 350° and $\nu t = 0°$ to 170° at intervals of 10°, the values repeating with reversed sign when θ_1 and νt are increased by 180°. This table, therefore, gives the ratio of the convection current to the emission current (multiplied by 1000) for a fine stream emitted from the filament at any assigned angle. The present method is more general than that of § 5, and can be applied to undisturbed orbits of any form when electrons are emitted at a constant rate from all the surface elements of the filament, provided that they are all caught when they return. Treating the columns of the table by Simpson's rule, we have the corresponding ratio for a filament emitting uniformly; numbers which are given in the last row on p. 224, below the main entries. These currents differ considerably from those of § 5. The difference is due to three main causes. Firstly, there is the approximation of § 4, where the range of θ_1 for capture by the anode was taken to be 180° for all values of ωt_1. To see what error it introduces, I calculated the entries under $\omega t = 0°$ and 90° in the table of § 5 by the new method and found $-0{\cdot}209$ and $0{\cdot}103$ instead of $-0{\cdot}222$ and $0{\cdot}096$. The error is thus of the order of 7 per cent. The remaining difference is due partly to precession and partly to the strong field near the edges of the sectors, which alters the limits of νt_1 for capture by the anode, especially near $\theta_1 = 90°$. A magnetron with a narrow gap will differ appreciably from one with a wide gap. Since the gap is often quite wide, it may actually be nearer to the truth to suppose that the disturbing field is uniform, though no accurate theory is possible for a wide gap. The fundamental term in the convection current taken from the last table is the real part of

$$(-0{\cdot}204-0{\cdot}191i)I\exp i\nu t, \tag{39}$$

instead of $(-0{\cdot}233-0{\cdot}098i)I\exp i\omega t$ in equation (15). The coefficient $0{\cdot}233$ in equation (18) is also replaced by $0{\cdot}204$. With all these

$1000\,(f_2 - f_1)$

θ_1 \ νt	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°	110°	120°	130°	140°	150°	160°	170°
0°	2	6	14	26	43	56	95	132	177	250	305	360	419	458	485	498	501	500
10°	0	2	6	14	25	42	69	103	151	212	282	354	414	460	490	505	512	509
20°	−12	0	3	8	19	37	58	90	138	217	293	373	439	487	513	527	530	525
30°	−28	−6	−1	5	14	29	59	93	138	235	335	425	490	531	550	560	556	546
40°	−47	−18	−1	3	11	31	55	104	192	302	424	512	565	590	600	599	590	571
50°	−69	−54	−4	1	10	27	62	146	267	442	561	615	658	653	650	641	621	599
60°	−96	−57	−15	1	7	30	87	241	522	654	703	720	721	714	700	680	657	625
70°	−115	−70	−29	0	9	45	232	689	785	802	799	794	780	765	750	717	686	653
80°	−131	−90	−45	−4	−1	451	892	901	892	880	863	847	826	804	780	754	715	680
90°	−159	−112	−63	−18	491	480	467	456	435	418	399	380	359	335	309	277	241	204
100°	−181	−137	−82	−35	0	−5	−15	−28	−41	−58	−75	−95	−115	−140	−165	−197	−232	−275
110°	−204	−156	−104	−56	12	−1	−9	−15	−26	−43	−66	−80	−93	−115	−142	−174	−209	−249
120°	−231	−185	−131	−81	−31	−2	−2	−8	−15	−27	−40	−55	−74	−93	−120	−147	−182	−223
130°	−270	−221	−170	−118	−70	−29	−1	−3	−9	−15	−25	−36	−54	−70	−94	−119	−150	−187
140°	−324	−275	−231	−180	−126	−80	−40	−9	2	−6	−12	−19	−30	−43	−60	−81	−108	−138
150°	−397	−355	−320	−280	−225	−170	−120	−73	−37	−10	0	1	−8	−15	−23	−39	−57	−77
160°	−458	−438	−417	−386	−358	−303	−251	−195	−140	−90	−50	−23	−5	0	−1	−6	−15	−25
170°	−490	−475	−460	−444	−425	−396	−357	−309	−252	−183	−136	−86	−48	−21	−5	0	−1	−4

$1000(f_2-f_1)$ (*cont.*)

θ_1 \ νt	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°	110°	120°	130°	140°	150°	160°	170°
180°	−498	−494	−486	−474	−457	−444	−405	−368	−323	−250	−195	−140	−81	−42	−15	−2	1	0
190°	−500	−498	−494	−486	−475	−458	−431	−397	−349	−288	−218	−146	−86	−40	−10	5	12	9
200°	−512	−500	−497	−492	−481	−463	−442	−410	−362	−283	−207	−127	−61	−13	13	27	30	25
210°	−528	−506	−501	−495	−486	−471	−441	−407	−362	−265	−165	−75	−10	31	50	60	56	46
220°	−547	−518	−501	−497	−489	−469	−445	−396	−308	−198	−76	12	65	90	100	99	90	71
230°	−569	−554	−504	−499	−490	−473	−438	−354	−233	−58	61	115	158	153	150	141	121	99
240°	−596	−557	−515	−499	−493	−470	−413	−259	22	154	203	220	221	214	200	180	157	125
250°	−615	−570	−529	−500	−491	−455	−268	189	285	302	299	294	280	265	250	217	186	153
260°	−631	−590	−545	−504	−501	−49	392	401	392	380	363	347	326	304	280	254	215	180
270°	−159	−112	−63	−18	491	480	467	456	435	418	399	380	359	335	309	277	241	204
280°	319	363	418	465	500	495	485	472	459	442	425	405	385	360	335	303	268	225
290°	296	344	396	444	488	499	491	485	474	457	434	420	407	385	358	326	291	251
300°	269	315	369	419	469	498	498	492	485	473	460	445	426	407	380	353	318	277
310°	230	279	330	382	430	471	499	497	491	485	475	464	446	430	406	381	350	313
320°	176	225	269	320	374	420	460	491	502	494	488	481	470	457	440	419	392	363
330°	103	145	180	220	275	330	380	427	463	490	500	501	492	485	477	461	443	423
340°	42	62	83	114	142	197	249	305	360	410	450	477	495	500	499	494	485	475
350°	10	25	40	56	75	104	143	191	248	317	364	414	452	479	495	500	499	496
	−192	−161	−128	−100	−41	5	47	113	162	207	238	255	274	286	276	263	247	221

differences, there is a good deal in hand, and theory predicts that oscillations will be maintained under ordinary conditions when all the electrons are caught after one excursion. It remains to see how subsequent orbits affect the numbers.

13. Calculation for multiple orbits

The cosine and sine integrals of § 11 are nearly equal when b/a is large. If they are put exactly equal, $\sin(\nu t_1-\theta_1)$ alone appears in the formulae and much of the work can be done with tables of single entry. Orbits which begin with the same value of $\nu t_1-\theta_1$ retain that value, and the values of α/a and β, throughout. Only a sketch of the calculations is given here.

We take as before $b/a = 500$, $V_0 = 225$, and $A = 5$. Then remembering that equations (32) to (34) give only amounts acquired in one orbit, we have approximately

$$c/a = 3{\cdot}14s, \qquad 1000B/V_0 = 22{\cdot}7s, \qquad 1000\beta = 5{\cdot}49s, \tag{40}$$

where s is the sum of the sines of the angles $\nu t_1-\theta_1$ for the successive orbits up to the beginning of a specified orbit, and c/a, B/V_0, β are the values at the end of that orbit. If the values (40) are placed in (24), it becomes a quadratic equation for s in terms of α/a. The two values of s are calculated for $\alpha/a = 1$ to $1{\cdot}4$ at intervals of $0{\cdot}02$, $1{\cdot}4$ to 2 at intervals of $0{\cdot}05$, and 2 to 12 at intervals of $0{\cdot}1$. Equation (25) then gives $k^{\frac{1}{2}}$ in terms of α/a. In this way a table of the positive and negative values of s is constructed for $k^{\frac{1}{2}} = 0$ to $1{\cdot}6$ at intervals of $0{\cdot}01$. It appears that $\alpha/a < 1$ when $0 < s < 0{\cdot}00230$, so that electrons are caught on the filament in that range, the corresponding limits of $k^{\frac{1}{2}}$ being 0 and $0{\cdot}0180$. A table of the filament deflexion $F = \pm(180°-2M)$ is constructed for the same values of $k^{\frac{1}{2}}$, and hence a table of F against s from $s = -2{\cdot}241$ through ∓ 0 to $1{\cdot}996$, F going from $29{\cdot}78°$ through $\pm 180°$ to $-26{\cdot}32°$. This table is not at equal intervals of s, but such a table would be inconvenient since F varies very quickly when s is small, and more manageable rates of change are obtained by taking $k^{\frac{1}{2}}$ as the variable of uniform increase. The formula for deciding whether an electron is caught on the anode in any orbit is

$$1000(r/b-1) = 5{\cdot}49s-22{\cdot}22P\cos\nu t_1-22{\cdot}22Q\sin\nu t_1, \tag{41}$$

where s stops short at the previous orbit and is zero for the first. Tables of $22{\cdot}22P$ and $22{\cdot}22Q$ for $\theta_1 = 0°$ to $180°$ at $1°$ intervals are

useful. Taking electrons for which $\theta_1 = 0°, 10°,...$ in the first orbit, tables of θ_1 and $r/b-1$ for the subsequent orbits are constructed at intervals of 10° in νt_1. Capture by the anode is slow, 5 per cent. of the electrons describing seven orbits, at which point the calculations are stopped. The tables show rapid variation. Few electrons are caught on the filament, but many come near it sooner or later, and every near approach means a large deflexion which may defer capture by several orbits. No reasonable amount of labour will give the current accurately, but a rough estimate is possible.

Instead of the table in § 12, in which $1000(f_2-f_1)$ is tabulated over finite ranges, we construct one in which the range is 10° in νt_1 with an assigned middle point; for example, the entry under $\nu t-\nu t_1 = 60°$ for any assigned θ_1 is the difference of the values of f for $\nu t-\nu t_1 = 55°$ and 65°. Consider the electrons emitted within 5° on either side of $\theta_1 = 0$ at times within 5° on either side of $\nu t_1 = 270°$. The electron $\theta_1 = 0$ has $\theta_1 = 55{\cdot}01°$ in the second orbit. Its charge is regarded, in our rough estimate, as spread over the appropriate range on either side of 55·01°, and then the table of $1000(f_2-f_1)$ gives the current due to the group of electrons at times $\nu t = 0°, 10°,....$ The mean electron is caught on the anode in the fifth orbit, and its fate is taken to decide that of the whole group; that is, the group is considered as describing the same number of half orbits as the mean electron. The process of calculating the current is straightforward, but precarious in view of the rapid variations referred to above. From the numbers obtained I extract those given on p. 227, in which the first line is copied from the table in § 12 (last line), and the others correspond to orbits after the first. The fundamental term in the convection current is the real part of

$$(0{\cdot}436-1{\cdot}928i)I\exp i\nu t,$$

and since the first term is positive the magnetron in these circumstances will not oscillate. The figures are very different from those for a single orbit. Whether the magnetron will maintain a range of potentials not including 5 volts can only be decided by detailed calculation.

14. Completion of the theory of an oscillating magnetron

Let the fundamental term in the convection current be the real part of $(a_1-ib_1)I\exp i\nu t$, where a_1 is negative. From § 6 the amplitude of potential maintained is $A = -a_1 IL/R(K+Cl\operatorname{cosec}^2\nu l/c)$.

1000 *convection current/emission current*

vt	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°	110°	120°	130°	140°	150°	160°	170°
1st orbit .	−192	−161	−128	−100	−41	5	47	113	162	207	238	255	274	286	276	263	247	221
2nd ,, .	−237	−108	13	166	270	349	409	472	570	624	645	657	645	627	564	500	438	341
3rd ,, .	58	153	262	361	458	543	594	615	614	615	609	597	413	278	240	179	94	12
4th ,, .	272	360	407	457	480	462	437	439	400	366	312	250	204	150	90	−1	−127	−188
5th ,, .	366	415	413	402	403	391	371	295	217	159	110	52	−15	−64	−127	−181	−257	−301
6th ,, .	159	181	190	180	125	53	−11	−25	−69	−125	−141	−161	−203	−235	−244	−184	−141	−144
7th ,, .	−70	−44	−46	−10	40	19	22	31	42	51	54	48	42	26	1	27	54	70
Total .	356	796	1111	1456	1735	1822	1869	1940	1936	1897	1827	1698	1360	1068	800	603	308	11

A value of A is assumed in calculating the current (cf. § 13), and the last equation gives I, the emission required to maintain that potential. A magnetron in which multiple orbits contribute appreciably to the current is different in principle from one in which they do not; for whereas a magnetron with one orbit yields to a comparatively simple linear theory, it is only the effect of the oscillatory field on a given orbit which is proportional to A when A/V_0 is small. When the electron approaches the filament it is deflected by an amount which varies rapidly with A, and the current in the second and subsequent orbits is not proportional to A, except possibly over a much smaller range.

As in § 4 of Note VIII,* the fundamental current need not have any particular phase relation with the fundamental voltage. All that is allowed for by the slight difference between the frequency of oscillation and the natural frequency. R being small, the harmonics are of smaller order.

15. Strength of oscillations of different frequency in a single-anode magnetron

We return in this section to a single-anode magnetron as simplified in Note VIII, taking the undisturbed electrons to describe circular orbits and the filament to capture them all on their return. Let the radius of the undisturbed cloud be exactly b, the radius of the anode. Then if $V = A\sin\nu t$, equation (3) of Note VIII gives

$$Q = A\sin(\nu t - \nu\pi/\omega).$$

A gap in the return stream of electrons begins at the anode at time π/ω, reaches the filament at time $2\pi/\omega$, and lasts for a time π/ν. In calculating current it is simpler to consider gaps rather than electrons, so that from equation (5) of Note VIII

$$i = K\frac{dV}{dt} - I\frac{\log(r_1/r_2)+\log(r_3/r_4)+\dots}{\log(b/a)},$$

where the gaps extend at time t from r_1 to r_2, r_3 to r_4, etc. The convection current is nearly equal to I when $r_1 = a$, that is when there is a gap at the filament, and nearly zero otherwise. Thus in each complete period, say $0 < \nu t < 2\pi$, there is a convection current I beginning when $\nu t = 2\nu\pi/\omega$ or an amount differing from it by a

* Loc. cit. 213.

multiple of 2π, and lasting for a half period. If this convection current is written

$$I\sum_{n=0}^{\infty}(C_n\cos n\nu t+D_n\sin n\nu t),$$

$\pi C_1 = -2\sin 2\nu\pi/\omega$ and $\pi D_1 = 2\cos 2\nu\pi/\omega$. We find, as in § 4 of Note VIII, that

$$A = -\frac{2ILC\cos 2\nu\pi/\omega}{\pi RK(K+C)}, \tag{42}$$

giving the amplitude of potential maintained in a condenser circuit. The condition of maintenance is that $\cos 2\nu\pi/\omega < 0$, so that ν/ω ranges from $\frac{1}{4}$ to $\frac{3}{4}$, $\frac{5}{4}$ to $\frac{7}{4}$, etc. Taking the factor $-\cos 2\nu\pi/\omega$ as the measure of the strength of an oscillation, the strength is greatest when $\nu/\omega = \frac{1}{2}$, $\frac{3}{2}$, etc.

16. Some concluding remarks

We have seen that there is reason for supposing that $\nu = \frac{1}{2}\omega$ is favourable for maintenance in an idealized single anode magnetron. This holds for any degree of grazing. It means that the time of the slowest oscillation maintained is about four times the time of transit of an electron from filament to anode, or twice the time of a complete excursion. But the resemblance to mechanical or electrical resonance is not very close, and there is no sharp maximum.

The application of the principle of energy to a problem that has already been solved is unnecessary, and I have not considered whether the principle of negative work can be improved by allowing for the attraction between the electrons and the charge which they induce in the plates. But it is instructive to consider the difference between low-frequency and high-frequency oscillations. The frequency may be much less than that at which the wave-length is comparable with the distance between the electrodes, and yet it may be quite wrong to reckon current by the number of electrons which fall on the electrodes. If so, it is idle to inquire on which electrode an electron falls, and thus some of the conclusions that are valid for low-frequency oscillations cease to hold when the frequency is high.

A TRANSFORMATION OF EULERIAN HYPERGEOMETRIC INTEGRALS

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1. It was shown by Euler that the hypergeometric series can be represented by the definite integrals

$$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt, \tag{1}$$

$$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}\,dt. \tag{2}$$

These integrals are convergent if, for example, we have $|x| < 1$ and $c > a$, $b > 0$. If the parameters a, b, c are any real or complex numbers such that the integrals are divergent, they can be replaced by contour integrals with the same integrands. The paths of integration, in the least favourable case, may be double circuits interlacing the singularities $t = 0$ and $t = 1$; and these can sometimes be replaced by simple circuits enclosing one or both of these points.

It was shown by Riemann that a hypergeometric function can be represented by forty-eight different Eulerian integrals, which can be reduced by an elementary change of the variable of integration to one or other of the forms (1) or (2). These cannot in general be thus transformed into one another. But Riemann conjectured, and Wirtinger subsequently proved, that both forms can be obtained by evaluating a certain multiple integral in two different ways. I can only mention in passing the striking integral representations of ordinary and generalized hypergeometric series recently obtained by Erdélyi. Wirtinger's and Erdélyi's methods both lie outside the scope of this note.

A few years ago I worked out the analogue of the Riemann-Wirtinger theory for the confluent hypergeometric function ${}_1F_1(a;c;x)$. This can also be represented by two different classes of Laplace integrals, which are derivable from the same double integral. But I found incidentally that the circumstances are exceptional when Kummer's confluent hypergeometric equation has a logarithmic

singularity; the two types of integrals can then be transformed directly into one another by integration by parts. I shall now prove that the Eulerian integrals of types (1) and (2), taken along appropriate contours, are also directly transformable into one another, when the ordinary hypergeometric equation has a logarithmic singularity.

2. If the Riemann equation corresponding to the scheme

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} \tag{3}$$

has a logarithmic singularity, we may assume without loss of generality that it lies at the point $x = 0$; for we may permute the three singularities by means of the linear transformations

$$x' = x, \qquad 1-x, \qquad \frac{1}{x}, \qquad \frac{1}{1-x}, \qquad \frac{x}{x-1}, \qquad \frac{x-1}{x}. \tag{4}$$

The exponents α, α' must differ by an integer; and, since they occur symmetrically in the equation, we assume $\alpha-\alpha' = n \geqslant 0$. We now put $y = x^{\alpha}(1-x)^{\gamma}y'$ to reduce (3) to the standard hypergeometric scheme

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & x \\ 1-c & b & c-a-b & \end{Bmatrix}, \tag{5}$$

where $1-c = \alpha'-\alpha = -n$ is a negative integer or zero. Thus c is a positive integer, while a, b are any real or complex numbers, subject only to the condition

$$a, b \neq 1, 2, 3, \ldots, c-1; \tag{6}$$

it is well known that, for these forbidden values, $x = 0$ is *not* a logarithmic singularity of the hypergeometric equation. The integral representation of type (2) is given by a simple contour integral

$$\begin{aligned} I &= \int_A^{(1+,0+)} (1-xt)^{-b} t^{a-1} (1-t)^{c-a-1}\, dt \\ &= (1-e^{-2i\pi a}) \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F(a, b; c; x), \end{aligned} \tag{7}$$

whose starting-point A lies in the real interval $0 < t < 1$, and where we have initially

$$\operatorname{am}(t) = 0, \qquad \operatorname{am}(1-t) = 0, \qquad |\operatorname{am}(1-xt)| < \pi.$$

If a is an integer not belonging to the forbidden set, the only singularity of the integrand is a pole at one or other of the points $t = 0, 1$. The path can then be contracted to a point and the integral evaluated as a Cauchy residue. The appropriate modifications of what follows for this case are left to the reader.

3. If b is not a forbidden integer (6), we transform (7) by $(c-1)$ successive integrations by parts; the integrated terms cancel around the contour, leaving

$$I = \frac{\Gamma(1-b)}{\Gamma(c-b)} x^{1-c} \int_A^{(1+,0+)} (1-xt)^{c-b-1} \frac{d^{c-1}}{dt^{c-1}} [t^{a-1}(1-t)^{c-a-1}]\, dt. \tag{8}$$

But (if a is not a forbidden integer) we have

$$\begin{aligned} \frac{d^{c-1}}{dt^{c-1}} [t^{a-1}(1-t)^{c-a-1}] &= \frac{d^{c-1}}{dt^{c-1}} t^{a-1} F(a-c+1, 1; 1; t) \\ &= \frac{\Gamma(a)}{\Gamma(a-c+1)} t^{a-c} F(a, 1; 1; t) \\ &= \frac{\Gamma(a)}{\Gamma(a-c+1)} t^{a-c}(1-t)^{-a}. \end{aligned} \tag{9}$$

Hence

$$I = \frac{\Gamma(1-b)\Gamma(a)}{\Gamma(c-b)\Gamma(a-c+1)} x^{1-c} \int_A^{(1+,0+)} (1-xt)^{c-b-1} t^{a-c} (1-t)^{-a}\, dt.$$

We now deform the contour into a circle C, $|t| = r$, where

$$1 < r < |1/x|,$$

and find

$$I = \frac{\Gamma(1-b)\Gamma(a)}{\Gamma(c-b)\Gamma(a-c+1)} e^{-i\pi a} x^{1-c} \int_C (1-xt)^{c-b-1} t^{a-c} (t-1)^{-a}\, dt, \tag{10}$$

where $\text{am}(t) = 0$, $\text{am}(t-1) = 0$, $|\text{am}(1-xt)| < \pi$ at the starting-point on the positive real axis. We now write

$$s = 1/xt, \tag{11}$$

which transforms $t = 0, 1, 1/x, \infty$ respectively into $s = \infty, 1/x, 1, 0$. The positive circuit C about $t = 0$ and $t = 1$ becomes a positive circuit about $s = \infty$ and $s = 1/x$, which is topologically the same thing as a negative circuit about $s = 0$ and $s = 1$, say the circle C in

the s-plane described clockwise. Changing the sign and using the same circuit as before, we have accordingly

$$x^{1-c}\int_C (1-xt)^{c-b-1}t^{a-c}(t-1)^{-a}\,dt$$

$$= -x^{1-c}\int_C \left(1-\frac{1}{s}\right)^{c-b-1}\left(\frac{1}{xs}\right)^{a-c}\left(\frac{1-xs}{xs}\right)^{-a}\left(\frac{-ds}{xs^2}\right)$$

$$= +\int_C (s-1)^{c-b-1}s^{b-1}(1-xs)^{-a}\,ds \tag{12}$$

$$= e^{i\pi(b-c+1)}\int_A^{(1+,0+)} (1-xs)^{-a}s^{b-1}(1-s)^{c-b-1}\,ds, \tag{13}$$

where in (12) we take $\mathrm{am}(s) = 0$, $\mathrm{am}(s-1) = 0$, $|\mathrm{am}(1-xs)| < \pi$ at the starting-point on the positive real axis, and then deform the contour back again into the circuit used in (7). Since we have

$$\frac{\Gamma(1-b)\Gamma(a)e^{-i\pi a+i\pi(b-c+1)}}{\Gamma(c-b)\Gamma(a-c+1)} = \frac{\Gamma(a)\Gamma(c-a)\sin\pi(c-a)}{\Gamma(b)\Gamma(c-b)\sin\pi b}\times e^{i\pi(-a+b-c+1)}$$

$$= \frac{\Gamma(a)\Gamma(c-a)(1-e^{-2i\pi a})}{\Gamma(b)\Gamma(c-b)(1-e^{-2i\pi b})}, \tag{14}$$

because c is an integer, we now get from (7) and (13) the relation

$$\int_A^{(1+,0+)} (1-xs)^{-a}s^{b-1}(1-s)^{c-b-1}\,ds$$

$$= \frac{(1-e^{-2i\pi b})\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F(a,b;c;x). \tag{15}$$

We have thus derived the representation of type (1) by a direct transformation of the type (2).

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LINEAR PARTIAL DIFFERENTIAL EQUATIONS (I)

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1. An equation of Darboux's

Under the term 'harmonic equations' Darboux* has discussed a class of equations which includes the particular equation

$$\frac{\partial^2 V}{\partial x \partial y} = \left\{\frac{m_1(m_1+1)}{(x+y)^2} - \frac{m_2(m_2+1)}{(x-y)^2} - \frac{m_3(m_3+1)}{(1-xy)^2} - \frac{m_4(m_4+1)}{(1+xy)^2}\right\} V. \quad (1)$$

For this equation he gave the solution (symbolic unless m_1, m_2, m_3, m_4 are positive integers)

$$\left(\frac{\partial}{t_1 \partial t_1}\right)^{m_1} \left(\frac{\partial}{t_2 \partial t_2}\right)^{m_2} \left(\frac{\partial}{t_3 \partial t_3}\right)^{m_3} \left(\frac{\partial}{t_4 \partial t_4}\right)^{m_4} \frac{(t_3+t_4)^{\Sigma m-4}}{t_1 t_2 t_3 t_4} \left\{\phi\left(\frac{t_1+t_2}{t_3+t_4}\right) + \psi\left(\frac{t_1-t_2}{t_3+t_4}\right)\right\}, \quad (2)$$

where $t_1 \equiv x+y$, $t_2 \equiv x-y$, $t_3 \equiv 1-xy$, $t_4 \equiv 1+xy$ and ϕ, ψ are arbitrary functions. In a formal sense at least this could claim to represent the 'general solution'. My object here is to give the *fundamental solution*† in terms of which both the general solution and the solution of Cauchy's problem are at once obtainable by well-known methods,‡ since the equation (1) is of Riemann's type

$$s+\alpha p+\beta q+\gamma z = 0.$$

Such a fundamental solution $U(X, Y; x, y)$ must have the properties that (i) it is a solution of the equation (1) (which is a self-adjoint equation), (ii) it reduces to unity along each of the two characteristics $x = X$ and $y = Y$. On this function of the four arguments x, y, X, Y impose the transformation

$$\left.\begin{aligned} x_1 &\equiv -\frac{(x-X)(y-Y)}{(x+y)(X+Y)}, \qquad & x_2 &\equiv \frac{(x-X)(y-Y)}{(x-y)(X-Y)} \\ x_3 &\equiv -\frac{(x-X)(y-Y)}{(1-xy)(1-XY)}, \qquad & x_4 &\equiv \frac{(x-X)(y-Y)}{(1+xy)(1+XY)} \end{aligned}\right\} \quad (3)$$

and write $$\delta_r \equiv x_r \frac{\partial}{\partial x_r} \quad (r = 1, 2, 3, 4).$$

* Darboux (**2**), livre 4, chap. **9**; in particular 226 (§ 415) equation (47). I have modified the notation slightly for my own purposes.

† Otherwise the 'Green's function' or the 'Riemann's function'.

‡ See, for instance, Darboux (**2**), livre 4, chap. 4 (in particular §§ 358–9); Goursat (**3**), chap. 26 (ii), § 498.

We then have the equivalence of operators

$$\frac{\partial}{\partial x}=\frac{1}{x-X}\left\{\frac{X+y}{x+y}\delta_1+\frac{X-y}{x-y}\delta_2+\frac{1-Xy}{1-xy}\delta_3+\frac{1+Xy}{1+xy}\delta_4\right\},$$

$$\frac{\partial}{\partial y}=\frac{1}{y-Y}\left\{\frac{x+Y}{x+y}\delta_1+\frac{x-Y}{x-y}\delta_2+\frac{1-xY}{1-xy}\delta_3+\frac{1+xY}{1+xy}\delta_4\right\}.$$

In forming the product of operators $\partial^2/\partial x\partial y$ we observe that the coefficient of δ_1^2 is

$$\frac{(X+y)(x+Y)}{(x-X)(y-Y)(x+y)^2}=\frac{1}{(x+y)^2}\left(1-\frac{1}{x_1}\right), \quad \text{and so on;}$$

and that the coefficient of $\delta_1\delta_2$ is

$$\frac{(X+y)(x-Y)+(X-y)(x+Y)}{(x-X)(y-Y)(x^2-y^2)}$$

$$=\frac{(x-y)(X+Y)+(x+y)(X-Y)}{(x-X)(y-Y)(x^2-y^2)}$$

$$=-\frac{1}{x_1(x+y)^2}+\frac{1}{x_2(x-y)^2}, \quad \text{and so on.}$$

There is also a first-order term, namely

$$\frac{1}{(x+y)^2}\delta_1-\frac{1}{(x-y)^2}\delta_2+\frac{1}{(1-xy)^2}\delta_3-\frac{1}{(1+xy)^2}\delta_4.$$

On collecting these results, substituting in (1), and rearranging we obtain in virtue of (i), the equation for U

$$\begin{aligned}\frac{1}{(x+y)^2}&\left[\delta_1^2+\delta_1-\frac{1}{x_1}\delta_1(\delta_1+\delta_2+\delta_3+\delta_4)-m_1(m_1+1)\right]U-\\ &-\frac{1}{(x-y)^2}\left[\delta_2^2+\delta_2-\frac{1}{x_2}\delta_2(\delta_1+\delta_2+\delta_3+\delta_4)-m_2(m_2+1)\right]U+\\ &+\frac{1}{(1-xy)^2}\left[\delta_3^2+\delta_3-\frac{1}{x_3}\delta_3(\delta_1+\delta_2+\delta_3+\delta_4)-m_3(m_3+1)\right]U-\\ &-\frac{1}{(1+xy)^2}\left[\delta_4^2+\delta_4-\frac{1}{x_4}\delta_4(\delta_1+\delta_2+\delta_3+\delta_4)-m_4(m_4+1)\right]U\\ &=0. \qquad (4)\end{aligned}$$

We can check the symmetry of this result by noticing that, if we change the signs of y, Y in (1), (3), then the suffixes 1, 2, 3, 4 become 2, 1, 4, 3; if we interchange y, Y with $1/y$, $1/Y$ respectively, the suffixes 1, 2, 3, 4 become 4, 3, 2, 1.

Evidently U will satisfy the equation (4) if it simultaneously satisfies the four equations

$$\delta_r(\delta_1+\delta_2+\delta_3+\delta_4)U = x_r(\delta_r-m_r)(\delta_r+m_r+1)U \quad (r = 1, 2, 3, 4). \tag{5}$$

In virtue of (ii) we look for a common solution of this system which reduces to unity when x_1, x_2, x_3, x_4 are all zero. Such a solution is given by the multiple hypergeometric series*

$$U = \sum_{r_1,\ldots,r_4=0}^{\infty} \frac{\prod_{s=1}^{4} (-m_s)_{r_s}(m_s+1)_{r_s}\dfrac{x_s^{r_s}}{r_s!}}{(r_1+r_2+r_3+r_4)!}, \tag{6}$$

where, as usual, $(t)_r$ denotes $t(t+1)\ldots(t+r-1)$. This series converges for sufficiently small x_s, i.e. in some neighbourhood of $x = X$, $y = Y$.

2. Some expressions for the function 八

This function (6) has some associations with Legendre's function P_n. To exhibit its four arguments x_s with their associated parameters m_s I introduce the notation†

$$U = 八_4\begin{bmatrix} m_1 & m_2 & m_3 & m_4 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix}.$$

If an x, say x_4 (or its associated m) vanishes, the series reduces to the corresponding *triple* hypergeometric series, which I shall denote by

$$八_3\begin{bmatrix} m_1 & m_2 & m_3 \\ x_1 & x_2 & x_3 \end{bmatrix},$$

similarly defining $八_2$ and $八_1$. I shall for brevity write

$$[m]_r \equiv (-m)_r(m+1)_r.$$

Then

$$八_1\begin{bmatrix} m \\ x \end{bmatrix} = \sum_{r=0}^{\infty} \frac{[m]_r x^r}{(r!)^2}$$

$$= {}_2F_1(-m, m+1; 1; x]$$

$$= P_m(1-2x), \tag{7}$$

by Murphy's formula.‡

* Multiple hypergeometric series in two variables (Appell's hypergeometric functions) and their associated partial differential equations are discussed by W. N. Bailey (**1**), chap. 9.

† This Chinese character 八 may be pronounced (and written) *pa*.

‡ E. T. Whittaker and G. N. Watson (**4**), § 15.22.

Again $$\Lambda_2\begin{bmatrix} m & n \\ x & y \end{bmatrix} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{[m]_r[n]_s x^r y^s}{r!\,s!\,(r+s)!}.$$

Now $$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} = \sum_{r=0}^{\infty}[\ldots]_{s=0}+\sum_{r=0}^{\infty}\sum_{s=1}^{\infty}.$$

The single sum gives $P_m(1-2x)$ as above. The double sum can be written

$$\sum_{r=0}^{\infty}\sum_{s=1}^{\infty}\frac{[m]_r[n]_s x^r y^s}{(r!)^2 s!(s-1)!}\,\frac{r!(s-1)!}{(r+s)!}$$

$$= \sum_{r=0}^{\infty}\sum_{s=1}^{\infty}\frac{[m]_r[n]_s x^r y^s}{(r!)^2 s!(s-1)!}\int_0^1 (1-t)^r t^{s-1}\,dt$$

$$= \int_0^1 \left\{\sum_{r=0}^{\infty}\frac{[m]_r(x-xt)^r}{(r!)^2}\right\}\frac{\partial}{\partial t}\left\{\sum_{s=0}^{\infty}\frac{[n]_s(yt)^s}{(s!)^2}\right\}dt$$

within the region of uniform convergence. Thus

$$\Lambda_2\begin{bmatrix} m & n \\ x & y \end{bmatrix} = P_m(1-2x)-2y\int_0^1 P_m(1-2x+2xt)P_n'(1-2yt)\,dt. \quad (8)$$

To put this into a form exhibiting the symmetry we remark that

$$P_m(1-2x) = [(t-1)P_m(1-2x+2xt)P_n(1-2yt)]_0^1$$

$$= \int_0^1 \{P_m P_n - 2x(1-t)P_m' P_n + 2y(1-t)P_m P_n'\}\,dt,$$

omitting the arguments of P_m, P_n for brevity. Hence

$$\Lambda_2\begin{bmatrix} m & n \\ x & y \end{bmatrix} = \int_0^1 \{P_m P_n - 2x(1-t)P_m' P_n - 2ytP_n P_n'\}\,dt,$$

i.e.

$$\Lambda_2\begin{bmatrix} m & n \\ x & y \end{bmatrix} = \left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+1\right)\int_0^1 P_m(1-2x+2xt)P_n(1-2yt)\,dt. \quad (9)$$

For Λ_4 I use the Dirichlet integral*

$$\frac{r_1!\,r_2!\,r_3!\,r_4!}{(r_1+r_2+r_3+r_4+3)!} = \int\cdots\int\prod_{s=1}^{4}(t_s^{r_s}\,dt_s), \quad (10)$$

* See, for instance, E. T. Whittaker and G. N. Watson (**4**), 258.

where the integration is taken over all non-negative values of the t_s such that $t_1+t_2+t_3+t_4 \leqslant 1$. Writing further

$$\Delta \equiv x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}+x_4\frac{\partial}{\partial x_4}, \tag{11}$$

we have

$$\Lambda_4\begin{bmatrix} m_1 & m_2 & m_3 & m_4 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix}$$
$$= (\Delta+1)(\Delta+2)(\Delta+3)\sum_{r_1,\dots,r_4=0}^{\infty}\left\{\prod_{s=1}^{4}\frac{[m]_{r_s}x_s^{r_s}}{(r_s!)^2}\right\}\frac{r_1!\,r_2!\,r_3!\,r_4!}{(r_1+r_2+r_3+r_4+3)!}.$$

Substituting from (10) and summing across the integral signs we get at length

$$\Lambda_4\begin{bmatrix} m_1 & m_2 & m_3 & m_4 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix}$$
$$= (\Delta+1)(\Delta+2)(\Delta+3)\iiiint P_{m_1}(1-2x_1t_1)P_{m_2}(1-2x_2t_2)\times$$
$$\times P_{m_3}(1-2x_3t_3)P_{m_4}(1-2x_4t_4)\,dt_1\,dt_2\,dt_3\,dt_4, \tag{12}$$

where the integral is extended over all non-negative values of t_s such that $t_1+t_2+t_3+t_4 \leqslant 1$.

As an alternative form to (12) we can write *symbolically*,

$$\Lambda_4\begin{bmatrix} m_1 & m_2 & m_3 & m_4 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \prod_{s=1}^{4}\left\{1-\int P'_{m_s}(1-2x_st_s)2x_s\,dt_s\right\}, \tag{13}$$

where in the expansion of the product every integral is extended over all non-negative values of t such that $\sum t \leqslant 1$. This is an extension of (8) and is proved similarly.

3. Some identities

If in the differential equation (1) we put $m_1 = m_2$ and $m_4 = m_3$, the equation reduces to

$$\frac{\partial^2 V}{4xy\,\partial x\partial y} = -\frac{m_2(m_2+1)}{(x^2-y^2)^2}+\frac{m_3(m_3+1)}{(1-x^2y^2)^2},$$

i.e.
$$\frac{\partial^2 V}{\partial x'\partial y'} = -\frac{m_2(m_2+1)}{(x'-y')^2}+\frac{m_3(m_3+1)}{(1-x'y')^2}, \tag{14}$$

if $x' \equiv x^2$, $y' = y^2$. This is again of the form (1) with m_1, m_4 zero.

Writing similarly $X' \equiv X^2$, $Y' \equiv Y^2$, we further find that

$$x_1+x_2-x_1x_2 = \frac{(x^2-X^2)(y^2-Y^2)}{(x^2-y^2)(X^2-Y^2)} = \frac{(x'-X')(y'-Y')}{(x'-y')(X'-Y')} \equiv x_2' \text{ say;}$$

$$x_3+x_4-x_3x_4 = -\frac{(x^2-X^2)(y^2-Y^2)}{(1-x^2y^2)(1-X^2Y^2)}$$

$$= -\frac{(x'-X')(y'-Y')}{(1-x'y')(1-X'Y')} \equiv x_3' \text{ say.}$$

Thus the fundamental solution of (14) is

$$\Lambda_2\begin{bmatrix} m_2 & m_3 \\ x_2' & x_3' \end{bmatrix}.$$

Since, as can be proved without difficulty, the fundamental solution of an equation of Riemann's form $s+\alpha p+\beta q+\gamma z = 0$ is unique, we can identify the two fundamental solutions of (14), getting an identity of the form*

$$\Lambda_4\begin{bmatrix} p & p & q & q \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \Lambda_2\begin{bmatrix} p & q \\ x_1+x_2-x_1x_2 & x_3+x_4-x_3x_4 \end{bmatrix}, \qquad (15)$$

Again, if in (14) we put $m_3 = m_2$, the equation can be written

$$\frac{\partial^2 V}{\partial x'\partial y'} = -\frac{m_2(m_2+1)(1-1/x'^2)(1-1/y'^2)}{\{(x'+1/x')-(y'+1/y')\}^2},$$

i.e.

$$\frac{\partial^2 V}{\partial x''\partial y''} = -\frac{m_2(m_2+1)}{(x''-y'')^2}, \qquad (16)$$

where $x'' \equiv x'+1/x'$, $y'' \equiv y'+1/y'$. This again has the form of (14) with m_3 zero. We find that

$$x_2'+x_3'-x_2'x_3' = \frac{(x''-X'')(y''-Y'')}{(x''-y'')(X''-Y'')} \equiv x_2'' \text{ say.}$$

Thus further

$$\Lambda_2\begin{bmatrix} m & m \\ x & y \end{bmatrix} = \Lambda_1\begin{bmatrix} m \\ x+y-xy \end{bmatrix} = P_n\{2(1-x)(1-y)-1\}, \qquad (17)$$

by (7). If we combine this with (15) we get for the Λ_4 with four equal parameters

$$\Lambda_4\begin{bmatrix} m & m & m & m \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix} = P_m\{2(1-x_1)(1-x_2)(1-x_3)(1-x_4)-1\}. \qquad (18)$$

* The function Λ_4 is, of course, symmetrical in its four parameters, and so (15) applies *mutatis mutandis* whenever these are equal in pairs.

If in these identities we substitute for Λ_4, etc., the various formulae of § 2, we get identities involving integrals of P_m: these I have not attempted to verify independently. The simplest of them comes by combining (8), (17), whence, writing $1-u$, $1-v$, $\frac{1}{2}(1+\tau)$ for x, y, t, we have

$$P_m(2uv-1)$$
$$= P_m(2u-1)-(1-v)\int\limits_{-1}^{1} P_m\{u+(1-u)\tau\}P'_m\{v-(1-v)\tau\}\,d\tau. \quad (19)$$

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